

ON THE CONTINUOUS EXTENSION OF KOBAYASHI ISOMETRIES

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(Communicated by Filippo Bracci)

ABSTRACT. We provide a sufficient condition for the continuous extension of isometries for the Kobayashi distance between bounded convex domains in complex Euclidean spaces having boundaries that are only slightly more regular than \mathcal{C}^1 . This is a generalization of a recent result by A. Zimmer.

1. INTRODUCTION

In this paper, we provide a sufficient condition for the continuous extension to $\overline{\Omega}_1$ of isometries, with respect to the Kobayashi distances on Ω_1 and Ω_2 , between a pair of bounded convex domains Ω_1 and Ω_2 in complex Euclidean spaces (of not necessarily the same dimension). In this setting it is well known that such isometries *do* exist. A consequence of fundamental work by Lempert [9, 10] is that if $\Omega \subset \mathbb{C}^n$ is a bounded convex domain, then given a pair of distinct points $z_1, z_2 \in \Omega$, there exists a *holomorphic* map $F : \mathbb{D} \rightarrow \Omega$ that is an isometry with respect to the Kobayashi distances on \mathbb{D} and Ω and such that $z_1, z_2 \in F(\mathbb{D})$. We call such a map a *complex geodesic* of Ω through z_1 and z_2 .

The question of whether a complex geodesic extends continuously to $\overline{\mathbb{D}}$ is not an easy one. The earliest result in this direction was given by Lempert [9], which states that if $\Omega \subset \mathbb{C}^n$ is strongly convex with \mathcal{C}^k -smooth boundary, $k \geq 2$, then every complex geodesic $F : \mathbb{D} \rightarrow \Omega$ extends to a \mathcal{C}^{k-2} -smooth mapping on $\overline{\mathbb{D}}$ (by a \mathcal{C}^0 -smooth mapping we mean a continuous one). Since then, there have been a number of works dealing with the continuous (or smooth) extension of complex geodesics; see [1, 2, 11, 15].

While Lempert's result might suggest that the boundary regularity of the target convex domain Ω controls the boundary behaviour of a complex geodesic of Ω , that is not the case; see [8, Remark 1.8] and [2, Example 1.2]. The latter example shows that there exist \mathcal{C}^∞ -smoothly bounded convex domains having complex geodesics that do not extend continuously to $\overline{\mathbb{D}}$. In view of this, the question of \mathcal{C}^0 -extension of Kobayashi isometries *in general* is certainly a challenging one.

Received by the editors August 16, 2019, and, in revised form, September 2, 2019, and December 20, 2019.

2010 *Mathematics Subject Classification.* Primary 32F45, 32H40; Secondary 53C22.

Key words and phrases. Boundary regularity, convex domains, isometries, Kobayashi distance.

This work was supported by a scholarship from the Indian Institute of Science and by a UGC CAS-II grant (Grant No. F.510/25/CAS-II/2018(SAP-I)).

Before we state the main result of this paper, let us look at the motivations behind it. Our chief motivation is the following recent result by Zimmer:

Result 1.1 (Zimmer [15, Theorem 2.18]). Let $\Omega_j \subset \mathbb{C}^{n_j}$, $j = 1, 2$, be bounded convex domains with $\mathcal{C}^{1,\alpha}$ -smooth boundaries, where $\alpha \in (0, 1)$. Suppose that Ω_2 is \mathbb{C} -strictly convex. Let $F : \Omega_1 \rightarrow \Omega_2$ be an isometric embedding with respect to the Kobayashi distances. Then F extends to a continuous map $\tilde{F} : \overline{\Omega}_1 \rightarrow \overline{\Omega}_2$.

Recall that for a convex, \mathcal{C}^1 -smoothly bounded domain $\Omega \subset \mathbb{C}^n$ to be \mathbb{C} -strictly convex means that for every $\xi \in \partial\Omega$,

$$(\xi + T_\xi^{\mathbb{C}}(\partial\Omega)) \cap \overline{\Omega} = \{\xi\},$$

where $T_\xi^{\mathbb{C}}(\partial\Omega)$ denotes the complex tangent space to $\partial\Omega$ at ξ , given by $T_\xi(\partial\Omega) \cap iT_\xi(\partial\Omega)$ and where we view $T_\xi(\partial\Omega)$ extrinsically as a real hyperplane in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ (also see Section 3).

A close reading of the proof of the above result reveals that it actually establishes a stronger result. Before we can state this result, we need to fix some pieces of notation. The first set of notation pertains to the real category. For $U \subset \mathbb{R}^d$ an open set and $f : U \rightarrow \mathbb{R}$ a \mathcal{C}^1 -smooth function, Df will denote the total derivative of f ; it is a continuous mapping from U into $\mathcal{L}(\mathbb{R}^d, \mathbb{R})$. For a vector $v \in \mathbb{R}^d$ with $\|v\| = 1$, D_v will denote the directional derivative in the direction of v .

In what follows, we shall identify \mathbb{C}^n with \mathbb{R}^{2n} in the following manner:

$$\mathbb{C}^n \ni z = (z_1, \dots, z_n) \longleftrightarrow (\operatorname{Re}(z_1), \operatorname{Im}(z_1), \operatorname{Re}(z_2), \operatorname{Im}(z_2), \dots, \operatorname{Re}(z_n), \operatorname{Im}(z_n)) \in \mathbb{R}^{2n}.$$

We let \mathbb{J} denote multiplication by i in \mathbb{C}^n regarded as an \mathbb{R} -linear map from \mathbb{C}^n to itself. In terms of the above identification,

$$\mathbb{J}(x_1, \dots, x_{2n}) = (-x_2, x_1, \dots, -x_{2n}, x_{2n-1}) \quad \forall (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}.$$

Given $a \in \mathbb{C}^n$ and $r > 0$, $B^{(n)}(a, r)$ will denote the open Euclidean ball in \mathbb{C}^n with centre a and radius r .

We are now in a position to state the above-mentioned result. In this result, for any $\xi \in \partial\Omega$, η^ξ will denote the unit inward-pointing normal to $\partial\Omega$ at ξ .

Result 1.1' (Follows from the proof of [15, Theorem 2.18]). Let $\Omega_j \subset \mathbb{C}^{n_j}$, $j = 1, 2$, be bounded convex domains with \mathcal{C}^1 -smooth boundaries. Suppose that there exist a constant $r > 0$, an $\alpha \in (0, 1)$, and, for each $j = 1, 2$, a defining function ρ_j for Ω_j such that for each $\xi \in \partial\Omega_j$, the directional derivative $D_{\mathbb{J}(\eta^\xi)}\rho_j$ is α -Hölder-continuous on the ball $B^{(n_j)}(\xi, r)$. If Ω_2 is \mathbb{C} -strictly convex, then every isometric embedding $F : \Omega_1 \rightarrow \Omega_2$ with respect to the Kobayashi distances extends to a continuous map $\tilde{F} : \overline{\Omega}_1 \rightarrow \overline{\Omega}_2$.

If Ω is \mathcal{C}^1 -smoothly bounded, $\partial\Omega \ni \xi \mapsto \mathbb{J}(\eta^\xi)$ is what is sometimes called the *complex-normal vector field* on $\partial\Omega$. The geometrical significance of the hypothesis in the above result is as follows: one does not require $\partial\Omega_j$ to be a $\mathcal{C}^{1,\alpha}$ -smooth manifold, $j = 1, 2$, for the conclusion of Result 1.1 to hold true; it suffices to control the behaviour of $\partial\Omega_1$ and $\partial\Omega_2$ in the complex-normal directions. As stated earlier, the proof of Result 1.1' follows from a careful reading of the proof of [15, Theorem 2.18] (and we shall see the required ingredients in the proof of our main theorem).

All of this raises the question whether the conclusion of the above results holds true under even lower regularity of $\partial\Omega_j$, $j = 1, 2$. This question is also suggested

by a related result in [2] in which certain convex domains with just \mathcal{C}^1 -smooth boundaries are considered (which we shall see below). This is the second motivation for our result. But first, we need a definition.

Definition 1.2. We say that a Lebesgue-measurable function $g : [0, \epsilon_0) \rightarrow [0, \infty)$, where $\epsilon_0 > 0$, satisfies a Dini condition if

$$\int_0^{\epsilon_0} \frac{g(t)}{t} dt < \infty.$$

Our main theorem (whose relation to Result 1.1, via Result 1.1', is clear) is:

Theorem 1.3. Let $\Omega_j \subset \mathbb{C}^{n_j}$, $j = 1, 2$, be bounded convex domains with \mathcal{C}^1 -smooth boundaries. Suppose that there exist a constant $r > 0$ and, for each $j = 1, 2$, a defining function ρ_j for Ω_j such that for each $\xi \in \partial\Omega_j$, the directional derivative $D_{\mathbb{J}(\eta\xi)}\rho_j$ has modulus of continuity ω on the ball $B^{(n_j)}(\xi, r)$. Assume that ω satisfies a Dini condition. If Ω_2 is \mathbb{C} -strictly convex, then every isometric embedding $F : \Omega_1 \rightarrow \Omega_2$ with respect to the Kobayashi distances extends to a continuous map $\tilde{F} : \overline{\Omega}_1 \rightarrow \overline{\Omega}_2$.

In view of our discussion on complex geodesics above, we have the following immediate corollary to Theorem 1.3:

Corollary 1.4. Let $\Omega \subset \mathbb{C}^n$ satisfy the conditions on Ω_2 of Theorem 1.3. Then every complex geodesic of Ω extends continuously to $\overline{\mathbb{D}}$.

We note that there exist many functions on intervals of the form $[0, \epsilon_0)$ that satisfy a Dini condition but which are not α -Hölder-continuous for any $\alpha \in (0, 1)$; examples are the functions

$$(1.1) \quad f_\epsilon(x) := \begin{cases} \frac{1}{|\log x|^{1+\epsilon}} & \text{if } x \in (0, 1), \\ 0 & \text{if } x = 0 \end{cases}$$

for arbitrary $\epsilon > 0$. While Theorem 1.3 generalizes Result 1.1, what is perhaps more suggestive are the geometric insights that its proof reveals. Firstly, given bounded convex domains Ω_1 and Ω_2 with \mathcal{C}^1 -smooth boundaries, given an isometry $F : \Omega_1 \rightarrow \Omega_2$ with respect to the Kobayashi distances, and given any point $\xi \in \partial\Omega_2$, how $\partial\Omega_2$ behaves in the *complex-tangential* directions is largely immaterial to the existence of a continuous extension of F to $\overline{\Omega}_1$, owing to adequate control on the local geometry of $\partial\Omega_2$ at ξ conferred by \mathbb{C} -strict convexity. Secondly, some elements of our proof reveal a certain bound for the Kobayashi distance that might be of independent interest. For greater clarity, Proposition 1.5 will present the above-mentioned bound for a special case (see Proposition 4.6 later for the more general result). We need a definition: we say that a domain $\Omega \subsetneq \mathbb{C}^n$ has $\mathcal{C}^{1, \text{Dini}}$ boundary if $\partial\Omega$ is, near each $\xi \in \partial\Omega$, the graph (relative to a coordinate chart around ξ) of a \mathcal{C}^1 function whose partial derivatives are Dini-continuous (i.e., have moduli of continuity that satisfy a Dini condition). With this definition, we have:

Proposition 1.5. Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain with $\mathcal{C}^{1, \text{Dini}}$ boundary. Let $z_0 \in \Omega$. Then, there exists a constant $C > 0$ such that

$$k_\Omega(z_0, z) \leq C + \frac{1}{2} \log \left(\frac{1}{\text{dist}(z, \Omega^\epsilon)} \right) \quad \forall z \in \Omega.$$

The above estimate is easy to deduce for domains with \mathcal{C}^2 -smooth boundaries. For domains with $\mathcal{C}^{1,\alpha}$ -smooth boundaries, it was established by Forstneric–Rosay [5]. In view of (1.1), Proposition 1.5 applies to domains that are not covered by [5].

We now state the result from [2] alluded to above. To state it, we need, given a bounded convex domain $\Omega \subset \mathbb{C}^n$ with \mathcal{C}^1 -smooth boundary, the notion of a function that *supports* Ω from the outside. Roughly speaking, such a function is a convex function $\Phi : (B^{(n-1)}(0, r_0), 0) \rightarrow ([0, \infty), 0)$ such that, for each $\xi \in \partial\Omega$, there exists a unitary change of coordinate $(\xi z_1, \dots, \xi z_n) \equiv (\xi z', \xi z_n)$ centred at ξ so that $\{\xi z_n = 0\} = T_\xi^\mathbb{C}(\partial\Omega)$ and such that a small open patch of $\partial\Omega$ around ξ lies on the convex side of the surface $\{(\xi z', \xi z_n) \in B^{(n-1)}(0, r_0) \times \mathbb{D} \mid \operatorname{Im}(\xi z_n) = \Phi(\xi z')\}$ (see [2, Definition 1.5]). Now, for an arbitrary $\alpha > 0$, let $\Psi_\alpha : [0, \infty) \rightarrow [0, \infty)$ be defined by

$$\Psi_\alpha(x) := \begin{cases} \exp(-1/x^\alpha) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

With these preparations, the result mentioned above is:

Result 1.6 (Bharali [2, Theorem 1.4]). Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain with \mathcal{C}^1 -smooth boundary. Suppose Ω is supported from the outside by a function of the form $\Phi(z') := \Psi_\alpha(\|z'\|)$, where $0 < \alpha < 1$. Then every complex geodesic of Ω extends continuously to $\overline{\mathbb{D}}$.

The above result has recently been extended to certain convex domains with non-smooth boundaries; see [3, Theorem 1.7]. The hypothesis of Result 1.6 is such that it admits domains Ω having boundary points that are not of finite type. As for the first four results in this section, their hypotheses *manifestly* cover the case where the domains involved have boundary points of infinite type. This is relevant because, by a result of Zimmer [14, Theorem 1.1] — given a bounded convex domain Ω with \mathcal{C}^∞ -smooth boundary and equipped with the Kobayashi distance k_Ω — if $\partial\Omega$ has points of infinite type, then (Ω, k_Ω) is *not* Gromov hyperbolic. Thus, *not only* is Theorem 1.3 (as is Result 1.1 or Result 1.1') a result involving domains with low boundary regularity, but it is one where (Ω_1, k_{Ω_1}) , (Ω_2, k_{Ω_2}) are not necessarily Gromov hyperbolic. That is, a very natural condition under which one may expect continuous extension to $\overline{\Omega}_1$ of $\Omega_1 \rightarrow \Omega_2$ Kobayashi isometries is unavailable, and this work is an inquiry into what other kinds of hypotheses suffice.

Results 1.6 and 1.1 both address the extension of complex geodesics and have apparently similar hypotheses. But neither subsumes the other. Also note that in Result 1.6 no constraints are placed on the way in which $\partial\Omega$ behaves in the *complex-normal* directions, but some degree of control is required in the complex-tangential directions. This is in stark contrast to Result 1.1 (or Result 1.1') and to our theorem. These together suggest the following:

Conjecture 1.7. *Let Ω be a bounded convex domain that has \mathcal{C}^1 -smooth boundary and is \mathbb{C} -strictly convex. Then every complex geodesic of Ω extends continuously to $\overline{\mathbb{D}}$.*

With the techniques currently known, this seems to be difficult to prove. Theorem 1.3 may be seen as evidence in support of this conjecture.

Before closing this section, we must mention a recent result in a similar vein by Bracci–Gaussier–Zimmer [4, Corollary 1.6]. This result concerns the continuous extension of $\Omega_1 \rightarrow \Omega_2$ Kobayashi quasi-isometries that are homeomorphisms.

While this result involves no assumption on the boundary regularity of Ω_1 or Ω_2 , necessarily $\dim(\Omega_1) = \dim(\Omega_2)$. Furthermore (Ω_1, k_{Ω_1}) is required to be Gromov hyperbolic. Thus, in view of our remarks above, [4, Corollary 1.6] is quite different from Theorem 1.3.

The plan of this paper is as follows: in Section 2, we collect some preliminary results that are not immediately related to Theorem 1.3 but which will play a crucial role in its proof. In Section 3, we collect three relevant facts about convex domains in \mathbb{C}^n . In Section 4, we prove the propositions that enable Result 1.1 to be generalized to Theorem 1.3. The result of Zimmer that we generalize, which leads to Theorem 1.3, is [15, Proposition 4.3]: our generalization is Proposition 4.5. Finally, in Section 5, we provide the proof of Theorem 1.3. In all these sections, $\|\cdot\|$ will denote the Euclidean norm.

2. TECHNICAL PRELIMINARIES

In this section we present some results that play a supporting role in the proofs of the main results in Section 4 and, therefore, of our main theorem. The first result, by S.E. Warschawski, is the principal tool that enables us to deal with the low regularity of $\partial\Omega_1$ and $\partial\Omega_2$ in Theorem 1.3.

To state this result, we need to fix some terminology. Given a rectifiable arc Γ in \mathbb{C} , we say that Γ has a *continuously turning tangent* if there is a C^1 -smooth diffeomorphism $\gamma : I \rightarrow \Gamma$, where I is an interval. Note, in particular, that γ' is non-vanishing. Given that Γ has a continuously turning tangent, a *tangent angle* at any point $\zeta \in \Gamma$ refers to the smaller of the two angles determined by the intersection of $T_\zeta\Gamma$ with a **fixed** line ℓ in \mathbb{C} . While different choices of ℓ define different tangent-angle functions on Γ , the difference between the tangent angles, determined by some fixed ℓ , at two points $\zeta_1, \zeta_2 \in \Gamma$ depends *only* on ζ_1 and ζ_2 (and, of course, on Γ), i.e., is independent of ℓ . For this reason, in the following result, and in all applications of it, we shall use the phrase “the tangent angle” without any further comment. If Γ is a closed rectifiable Jordan curve in \mathbb{C} , analogous observations can be made about arc length. With these words, we can now state the following:

Result 2.1 ([13, Theorem 1]). Let C be a closed rectifiable Jordan curve in \mathbb{C} and let C have a continuously turning tangent in a C -open neighbourhood of a point $\zeta_0 \in C$. Suppose that the tangent angle $\tau(s)$ as a function of arc length s has a modulus of continuity ω at the point s_0 corresponding to ζ_0 — i.e., there exists a constant $\sigma > 0$ such that

$$(2.1) \quad |\tau(s) - \tau(s_0)| \leq \omega(|s - s_0|) \quad \text{whenever } |s - s_0| \leq \sigma$$

— that satisfies the following condition:

$$(2.2) \quad \int_0^\sigma \frac{\omega(t)}{t} dt < \infty.$$

Let f be a biholomorphic map of \mathbb{D} onto the region D enclosed by C and let $\zeta_0 = f(z_0)$. Then

$$\lim_{\mathbb{D} \ni z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0)$$

exists, and

$$\lim_{S \ni z \rightarrow z_0} f'(z) = f'(z_0)$$

for any Stolz angle S with vertex at z_0 . Furthermore, $f'(z_0) \neq 0$.

We refer the reader to [12, Chapter 1] for a definition of a Stolz angle in \mathbb{D} .

Remark 2.2. Note that since the ω appearing in the above result is a modulus of continuity, it is a non-decreasing function on $[0, \sigma]$ (and is continuous at 0). For this reason, the integrand in (2.2) is Lebesgue measurable. Secondly, in the statement of Result 2.1, we have tacitly used Carathéodory's theorem to conclude that, given that C is rectifiable, the map $f : \mathbb{D} \rightarrow D$ in the above result extends to a homeomorphism of $\overline{\mathbb{D}}$.

The following is an immediate corollary to the above result.

Corollary 2.3. *In the set-up described in Result 2.1, if g denotes f^{-1} , then*

$$(2.3) \quad L := \lim_{D \ni \zeta \rightarrow \zeta_0} \frac{g(\zeta) - z_0}{\zeta - \zeta_0}$$

exists and is non-zero.

We will also need the following simple lemma involving moduli of continuity.

Lemma 2.4. *Let f be a real-valued function defined on a ball $B^{(n)}(a, r) \subset \mathbb{C}^n$. Then the modulus of continuity ω of f on $B^{(n)}(a, r)$ is sub-additive; i.e., for all $s, t \in [0, 2r]$ such that $s + t < 2r$, $\omega(s + t) \leq \omega(s) + \omega(t)$.*

Proof. Suppose $x, y \in B^{(n)}(a, r)$ and $\|x - y\| \leq s + t$. If $\|x - y\| \leq s$, then $|f(x) - f(y)| \leq \omega(s) \leq \omega(s) + \omega(t)$. Now suppose that $\|x - y\| > s$. Note that

$$\left| f(x) - f\left(x + s \frac{y - x}{\|y - x\|}\right) \right| \leq \omega(s)$$

and

$$\begin{aligned} \left| f\left(x + s \frac{y - x}{\|y - x\|}\right) - f(y) \right| &= \left| f\left(x + s \frac{y - x}{\|y - x\|}\right) - f\left(x + \|y - x\| \frac{y - x}{\|y - x\|}\right) \right| \\ &\leq \omega(\|y - x\| - s) \leq \omega(t). \end{aligned}$$

Consequently, by the triangle inequality, $|f(y) - f(x)| \leq \omega(s) + \omega(t)$. Since x and y were arbitrary points in $B^{(n)}(a, r)$ satisfying $\|x - y\| \leq s + t$, it follows that $\omega(s + t) \leq \omega(s) + \omega(t)$. \square

3. SOME FACTS ABOUT CONVEX DOMAINS

In this section we record some facts about convex domains in \mathbb{C}^n . The first two were proved by Zimmer in [15]. All of them are needed in the proof of Theorem 1.3. The first result concerns a lower bound for the Kobayashi distance on arbitrary convex domains. First, some notation: in what follows, given a domain $\Omega \subset \mathbb{C}^n$, k_Ω will denote the Kobayashi pseudodistance on Ω , and κ_Ω will denote the Kobayashi pseudometric on Ω .

Result 3.1 ([15, Lemma 4.2]). Let $\Omega \subsetneq \mathbb{C}^n$ be a convex domain and let $H \subset \mathbb{C}^n$ be a complex affine hyperplane such that $\Omega \cap H = \emptyset$. Then, for every $z_1, z_2 \in \Omega$,

$$(3.1) \quad k_\Omega(z_1, z_2) \geq \frac{1}{2} \left| \log \left(\frac{\text{dist}(z_1, H)}{\text{dist}(z_2, H)} \right) \right|.$$

The next result is a sharper lower bound for the Kobayashi distance between a pair of points in a bounded convex domain with \mathcal{C}^1 -smooth boundary under an additional assumption. At this point, we wish to state a key clarification about our notation. Whenever $\Omega \subset \mathbb{C}^n$ is a \mathcal{C}^1 -smoothly bounded domain and $\xi \in \partial\Omega$, $(\xi + T_\xi^{\mathbb{C}}(\partial\Omega))$ will be understood to be a certain set in \mathbb{C}^n . $T_\xi(\partial\Omega)$ will denote

the real tangent space to $\partial\Omega$ at ξ viewed *extrinsically*, i.e., as a real hyperplane in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, taking into account that $\partial\Omega$ is \mathcal{C}^1 -smoothly embedded in \mathbb{C}^n . Then,

$$T_{\xi}^{\mathbb{C}}(\partial\Omega) := T_{\xi}(\partial\Omega) \cap iT_{\xi}(\partial\Omega),$$

with $T_{\xi}(\partial\Omega)$ being viewed extrinsically.

Result 3.2 ([15, Lemma 4.5]). Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain with \mathcal{C}^1 -smooth boundary. Let $\xi, \xi' \in \partial\Omega$ and suppose that $\xi + T_{\xi}^{\mathbb{C}}(\partial\Omega) \neq \xi' + T_{\xi'}^{\mathbb{C}}(\partial\Omega)$. Then there exist constants $\epsilon, C > 0$ such that for every $p \in \Omega$ with $\text{dist}(p, \xi + T_{\xi}^{\mathbb{C}}(\partial\Omega)) \leq \epsilon$ and every $q \in \Omega$ with $\text{dist}(q, \xi' + T_{\xi'}^{\mathbb{C}}(\partial\Omega)) \leq \epsilon$,

$$(3.2) \quad k_{\Omega}(p, q) \geq \frac{1}{2} \log \left(\frac{1}{\delta_{\Omega}(p)} \right) + \frac{1}{2} \log \left(\frac{1}{\delta_{\Omega}(q)} \right) - C.$$

Here, for any $z \in \Omega$, $\delta_{\Omega}(z) := \text{dist}(z, \Omega^c)$.

The following result provides bounds for the Kobayashi *metric* on convex domains.

Result 3.3 (Graham [6, Theorem 3]; also see [7]). Let $\Omega \subset \mathbb{C}^n$ be a convex domain. Given $p \in \Omega$ and $v \in T_p^{(1,0)}\Omega$, we let $r_{\Omega}(p, v)$ denote the supremum of the radii of the disks centred at p , tangent to v , and included in Ω . Then

$$(3.3) \quad \frac{\|v\|}{2r_{\Omega}(p, v)} \leq \kappa_{\Omega}(p, v) \leq \frac{\|v\|}{r_{\Omega}(p, v)}.$$

4. ESSENTIAL PROPOSITIONS

The goal of this section is to prove certain technical results that are essential for extending the scope of an idea in [15] to the sorts of domains considered in Theorem 1.3: specifically, that inward-pointing normals can be parametrized as K -almost-geodesics for some $K \geq 1$. In [15], this relies on a construction by Forstneric–Rosay in [5, Proposition 2.5] for estimating effectively the Kobayashi distance close to the boundary of a domain Ω whose boundary is of class $\mathcal{C}^{1,\alpha}$.

Definition 4.1 (Zimmer [15, Definition 3.2]). Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. For $K \geq 1$, by a K -almost geodesic in Ω (with respect to the Kobayashi distance) we mean a mapping $\sigma : I \rightarrow \Omega$, where I is an interval in \mathbb{R} , such that

- (1) $|s - t| - \log(K) \leq k_{\Omega}(\sigma(s), \sigma(t)) \leq |s - t| + \log(K) \quad \forall s, t \in I$ and
- (2) $k_{\Omega}(\sigma(s), \sigma(t)) \leq K|s - t| \quad \forall s, t \in I$.

The Forstneric–Rosay estimate involves embedding a certain compact planar set $\overline{\mathcal{D}}$ with $0 \in \partial\mathcal{D}$ into Ω so that its image osculates $\partial\Omega$ at the image of 0. Since the domains considered in Theorem 1.3 need not necessarily have boundaries of class $\mathcal{C}^{1,\alpha}$, we must modify *significantly* the constructions in [5, Proposition 2.5], starting with a class of planar domains better adapted to the domains Ω_1 and Ω_2 of Theorem 1.3.

Such a domain (which must contain 0 in its boundary) must have a defining function that is \mathcal{C}^1 near 0 whose derivative (while not necessarily α -Hölder-continuous for any $\alpha \in (0, 1)$) will have a modulus of continuity that satisfies a Dini condition. To this end, with ω as in Theorem 1.3, we define the function $h : (-2r, 2r) \rightarrow [0, \infty)$

as follows:

$$h(t) := \begin{cases} \int_t^0 \omega(-y)dy & \text{if } t < 0, \\ \int_0^t \omega(y)dy & \text{if } t \geq 0. \end{cases}$$

The following properties of h are easily verified: $h(0) = 0$, h is strictly increasing on $[0, 2r)$ and strictly decreasing on $(-2r, 0]$, and $h'(0) = 0$. Then, for $\alpha, \tau > 0$, consider the domain

$$\mathcal{D}(\alpha, \tau) := \{\zeta = s + it \in \mathbb{C} \mid |t| < \tau, \alpha h(t) < s < \tau\}.$$

The following property of the domains $\mathcal{D}(\alpha, \tau)$ is obvious from the definition: if $\zeta = s + it \in \mathcal{D}(\alpha, \tau)$, then $|t| \leq h^{-1}(s/\alpha)$. Near 0, a defining function for $\mathcal{D}(\alpha, \tau)$ is $\varrho(s, t) := \alpha h(t) - s$. Its total derivative at the point (s, t) , $(D\varrho)(s, t)$, with respect to the standard basis of \mathbb{R}^2 , is

$$[-1 \quad \alpha h'(t)].$$

It is easily checked that the modulus of continuity of $D\varrho$ at 0 is $\alpha\omega$: i.e., for every $t \in (-2r, 2r)$, $\|(D\varrho)(s, t) - (D\varrho)(0)\| = \alpha\omega(|t|)$.

We will use the following fact in our proof below: if $w \in \mathbb{C}^n$, then $\mathbb{J}(w)$ is orthogonal to w with respect to the standard real inner product on $\mathbb{R}^{2n} \longleftrightarrow \mathbb{C}^n$. With this remark, we now state and prove the following proposition.

Proposition 4.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain having the properties common to Ω_1 and Ω_2 as stated in Theorem 1.3. For $\xi \in \partial\Omega$, let $\Psi_\xi : \mathbb{C} \rightarrow \mathbb{C}^n$ denote the \mathbb{C} -affine map*

$$\Psi_\xi(\zeta) := \xi + \zeta\eta^\xi \quad \forall \zeta \in \mathbb{C}.$$

Then there exist constants $\alpha, \tau > 0$ such that, for every $\xi \in \partial\Omega$, $\Psi_\xi(\mathcal{D}(\alpha, \tau)) \subset \Omega$.

Proof. We are given a \mathcal{C}^1 defining function ρ defined on a neighbourhood U of $\partial\Omega$ and we are given an $r > 0$ such that, for every $\xi \in \partial\Omega$, the directional derivative $D_{\mathbb{J}(\eta^\xi)}\rho$ has on $B^{(n)}(\xi, r)$ modulus of continuity ω . We shall identify $\mathcal{L}(\mathbb{R}^{2n}, \mathbb{R})$ with \mathbb{R}^{2n} via the matrix representation of the elements of $\mathcal{L}(\mathbb{R}^{2n}, \mathbb{R})$ relative to the standard basis of \mathbb{R}^{2n} . Since $D\rho$ does not vanish on $\partial\Omega$, there is an $m > 0$ such that, for every $\xi \in \partial\Omega$, $\|(D\rho)(\xi)\| \geq m$. Furthermore, if we choose a neighbourhood V of $\partial\Omega$ in U such that \overline{V} is a compact subset of U , then $D\rho$ is uniformly continuous on V . In particular, there is a δ_0 , $0 < \delta_0 < r$, such that

$$(4.1) \quad \|(D\rho)(\xi) - (D\rho)(\xi')\| \leq m/2 \quad \forall \xi, \xi' \in V \text{ such that } \|\xi - \xi'\| \leq \delta_0.$$

Choose $\tau > 0$ so small that $\sqrt{2}\tau < \delta_0$; it then follows that

$$\{s + it \in \mathbb{C} \mid |t| < \tau, 0 < s < \tau\} \subset D(0, \delta_0).$$

Then, for any $\alpha > 0$, $\mathcal{D}(\alpha, \tau) \subset D(0, \delta_0)$. We fix a value of $\alpha \geq 1$ so large that $2/\alpha \leq m/4$. We shall soon see the reason for this choice. We may also need to shrink τ further. The precise value of τ that works will be presented below.

For the rest of the proof we fix $\xi \in \partial\Omega$ and $\zeta \in \mathcal{D}(\alpha, \tau)$. In what follows, $a + ib$ ($a, b \in \mathbb{R}^n$) will, for simplicity of notation, denote either a complex vector or the vector $(a_1, b_1, \dots, a_n, b_n) \in \mathbb{R}^{2n}$, the intended meaning being clear from the

context. By Taylor's theorem and writing $\zeta = s + it$ (and $\langle \cdot, \cdot \rangle$ denoting the usual inner product on \mathbb{R}^{2n}), we get

$$\begin{aligned}
 \rho(\xi + \zeta\eta^\xi) &= \rho(\xi) + (D\rho)(\xi)(\zeta\eta^\xi) + \int_0^1 ((D\rho)(\xi + x\zeta\eta^\xi) - (D\rho)(\xi))(\zeta\eta^\xi)dx \\
 &= s\langle \nabla\rho(\xi), \eta^\xi \rangle + t\langle \nabla\rho(\xi), \mathbb{J}(\eta^\xi) \rangle + s \int_0^1 ((D\rho)(\xi + x\zeta\eta^\xi) - (D\rho)(\xi))(\eta^\xi)dx \\
 &\quad + t \int_0^1 ((D\rho)(\xi + x\zeta\eta^\xi) - (D\rho)(\xi))(\mathbb{J}(\eta^\xi))dx \quad [\text{since } \rho(\xi) = 0] \\
 &\leq -sm + s \int_0^1 |((D\rho)(\xi + x\zeta\eta^\xi) - (D\rho)(\xi))(\eta^\xi)|dx \\
 (4.2) \quad &\quad + |t| \int_0^1 |(D_{\mathbb{J}(\eta^\xi)}\rho)(\xi + x\zeta\eta^\xi) - (D_{\mathbb{J}(\eta^\xi)}\rho)(\xi)|dx \quad [\text{since } \mathbb{J}(\eta^\xi) \perp \nabla\rho(\xi)].
 \end{aligned}$$

Since, for every $\xi \in \partial\Omega$, every $x \in [0, 1]$, and every $\zeta \in \mathcal{D}(\alpha, \tau)$, $\|(\xi + x\zeta\eta^\xi) - \xi\| = x|\zeta| < \delta_0$,

$$\|(D\rho)(\xi + x\zeta\eta^\xi) - (D\rho)(\xi)\| \leq \frac{m}{2},$$

by (4.1). Therefore the second term on the right hand side of (4.2) is less than or equal to $sm/2$. As for the third term, note that for every $x \in [0, 1]$,

$$|(D_{\mathbb{J}(\eta^\xi)}\rho)(\xi + x\zeta\eta^\xi) - (D_{\mathbb{J}(\eta^\xi)}\rho)(\xi)| \leq \omega(\|(\xi + x\zeta\eta^\xi) - \xi\|) = \omega(x|\zeta|).$$

So the third term on the right hand side of (4.2) is less than or equal to

$$|t| \int_0^1 \omega(x|\zeta|)dx.$$

Therefore we get, from (4.2),

$$(4.3) \quad \rho(\xi + \zeta\eta^\xi) \leq -sm + (sm/2) + |t| \int_0^1 \omega(x|\zeta|)dx.$$

Since $\zeta \in \mathcal{D}(\alpha, \tau)$,

$$(4.4) \quad |\zeta| \leq (s^2 + (h^{-1}(s/\alpha))^2)^{1/2} = h^{-1}(s/\alpha) \left(1 + \left(\frac{s}{h^{-1}(s/\alpha)} \right)^2 \right)^{1/2}.$$

Since $h'(0) = 0$, we have $\lim_{x \rightarrow 0+} x/h^{-1}(x) = 0$. Therefore, we can shrink τ so that

$$(4.5) \quad x/h^{-1}(x) \leq 1/\alpha \quad \forall x \in (0, \tau).$$

Now, from (4.4), the fact that $0 < s/\alpha < \tau$ (since $s/\alpha \leq s$ by our choice of α), and (4.5), we have

$$1 + \left(\frac{s}{h^{-1}(s/\alpha)} \right)^2 = 1 + \alpha^2 \left(\frac{s/\alpha}{h^{-1}(s/\alpha)} \right)^2 \leq 2.$$

From the last inequality and (4.4),

$$|\zeta| \leq \sqrt{2}h^{-1}(s/\alpha).$$

Using the above in (4.3) we get that

$$\begin{aligned}
 \rho(\xi + \zeta\eta^\xi) &\leq -(sm/2) + |t| \int_0^1 \omega\left(\sqrt{2}xh^{-1}(s/\alpha)\right) dx \\
 &\leq -(sm/2) + h^{-1}(s/\alpha) \int_0^1 \omega\left(2xh^{-1}(s/\alpha)\right) dx \\
 &\leq s \left(-\frac{m}{2} + \frac{2h^{-1}(s/\alpha)}{s} \int_0^1 \omega\left(xh^{-1}(s/\alpha)\right) dx \right) \\
 &= s \left(-\frac{m}{2} + \frac{2}{s} \int_0^{h^{-1}(s/\alpha)} \omega(u) du \right) \quad [\text{by change of variables}] \\
 &= s \left(-\frac{m}{2} + \frac{2}{\alpha} \right) \leq -\frac{sm}{4} < 0,
 \end{aligned}$$

by the choice of α discussed above. We note here that the third inequality follows from Lemma 2.4. Therefore, $\xi + \zeta\eta^\xi \in \Omega$. Since $\xi \in \partial\Omega$ and $\zeta \in \mathcal{D}(\alpha, \tau)$ were arbitrary, the proof is complete. \square

The proof of Theorem 1.3, as we shall see, relies crucially on the conclusion of Result 2.1, for the point $z_0 = 1$, when applied to the domains $\mathcal{D}(\alpha, \tau)$. We must therefore verify that the hypotheses of that result hold for $\mathcal{D}(\alpha, \tau)$. It is enough to show that the modulus of continuity of the tangent angle to $\partial\mathcal{D}(\alpha, \tau)$ near 0, regarded as a function of arc length, satisfies a Dini condition. Before we do this, we note the following elementary fact:

$$(4.6) \quad |\tan^{-1}(x)| \leq |x| \quad \forall x \in \mathbb{R}.$$

We also note that given α and τ , a parametrization of $\partial\mathcal{D}(\alpha, \tau)$ near 0 is given by $\Phi := y \mapsto (\alpha h(y), y) : [-\epsilon_0, \epsilon_0] \rightarrow \mathbb{R}^2$, where ϵ_0 is a suitably small positive quantity depending on α and τ . Therefore the tangent angle to $\partial\mathcal{D}(\alpha, \tau)$ near 0, as a function of y , is

$$(4.7) \quad \hat{\theta}(y) = \tan^{-1}(\alpha h'(y)) \quad \forall y \in [-\epsilon_0, \epsilon_0].$$

(In this instance, the line ℓ , as introduced in the explanations preceding Result 2.1, is the imaginary axis of \mathbb{C} .) Now we present the following lemma.

Lemma 4.3. *The tangent angle θ of $\partial\mathcal{D}(\alpha, \tau)$ near 0, regarded as a function of arc length, has a modulus of continuity that is dominated by $\alpha\omega$ (and therefore satisfies a Dini condition).*

Proof. First we determine the arc length as a function of y . We will reckon the (signed) arc length s from 0 and such that $s(x + iy) < 0$ for $x + iy \in \partial\mathcal{D}(\alpha, \tau)$ and $y < 0$, and $s(x + iy) > 0$ for $x + iy \in \partial\mathcal{D}(\alpha, \tau)$ and $y > 0$ (we are only interested in the arc length near 0). Using the parametrization Φ referred to just prior to (4.7), we see that the function that gives the arc length as a function of y , which we denote by G , is

$$(4.8) \quad G(y) = \int_0^y \|\Phi'(t)\| dt = \int_0^y [1 + \alpha^2 \omega(|t|)^2]^{1/2} dt$$

for all $y \in (-\epsilon_0, \epsilon_0)$. Clearly,

$$(4.9) \quad |G(y)| \geq |y| \quad \forall y \in (-\epsilon_0, \epsilon_0).$$

Note that G is a strictly increasing odd function on $(-\epsilon_0, \epsilon_0)$. So G^{-1} is a function that is defined on $(-G(\epsilon_0), G(\epsilon_0))$ and is strictly increasing. Taking $y = G^{-1}(s)$, $s \in (-G(\epsilon_0), G(\epsilon_0))$ in (4.9), we get

$$(4.10) \quad |G^{-1}(s)| \leq |s| \quad \forall s \in (-G(\epsilon_0), G(\epsilon_0)).$$

Now the function θ that gives the tangent angle as a function of arc length is

$$\theta(s) = \widehat{\theta}(G^{-1}(s)) \quad \forall s \in (-G(\epsilon_0), G(\epsilon_0)).$$

Recall that $|h'(y)| = \omega(|y|)$ and ω is continuous at 0. Thus, we may suppose that ϵ_0 is so small that for every $y \in (-\epsilon_0, \epsilon_0)$, $\alpha\omega(|y|) \leq 1$. Therefore, for an arbitrary $s \in (-G(\epsilon_0), G(\epsilon_0))$,

$$\begin{aligned} |\theta(s)| &= |\widehat{\theta}(G^{-1}(s))| = |\tan^{-1}(\alpha h'(G^{-1}(s)))| && \text{[by (4.7)]} \\ &\leq \alpha |h'(G^{-1}(s))| && \text{[by (4.6)]} \\ &\leq \alpha \omega(|s|) && \text{[by (4.10)].} \end{aligned}$$

This gives us the required result. \square

Remark 4.4. The significance of Lemma 4.3 is as follows: for every $\alpha > 0, \tau > 0$, the domain $\mathcal{D}(\alpha, \tau)$ satisfies the hypotheses of Result 2.1 at $0 \in \partial\mathcal{D}(\alpha, \tau)$. Thus, Corollary 2.3 holds.

We are now ready to state and prove a generalization of Proposition 4.3 in [15]. The generalization of the latter result alone suffices to yield a generalization of Theorem 2.11 in [15], which is fundamental to establishing an extension-of-isometries theorem.

Proposition 4.5. *Let Ω be an open convex subset of \mathbb{C}^n having the properties possessed in common by Ω_1 and Ω_2 in the statement of Theorem 1.3. Then there exist $K, \epsilon > 0$ such that for every $\xi \in \partial\Omega$,*

$$(4.11) \quad \sigma_\xi := t \mapsto \xi + \epsilon e^{-2t} \eta^\xi : [0, \infty) \rightarrow \Omega$$

is a K -almost-geodesic.

Proof. Our proof will resemble, in *essence*, the proof of Proposition 4.3 in [15]. The two proofs will differ in the key detail that we must work with the domains $\mathcal{D}(\alpha, \tau)$, which are adapted to the domain Ω under consideration.

By Proposition 4.2 there exist $\alpha, \tau > 0$ such that for every $\xi \in \partial\Omega$, $\xi + \mathcal{D}(\alpha, \tau)\eta^\xi \subset \Omega$. As $\mathcal{D}(\alpha, \tau)$ is a bounded open convex subset of \mathbb{C} symmetric about the real axis, there exists a biholomorphism $g : \mathcal{D}(\alpha, \tau) \rightarrow \mathbb{D}$ such that $g(\mathcal{D}(\alpha, \tau) \cap \mathbb{R}) = \mathbb{D} \cap \mathbb{R}$. By Carathéodory's theorem, g extends to a homeomorphism from $\overline{\mathcal{D}(\alpha, \tau)}$ to $\overline{\mathbb{D}}$. We may suppose, without loss of generality, that $g(0) = 1$. By the remark following the proof of Lemma 4.3, we see that we can apply Corollary 2.3 to g to conclude that

$$\lim_{\mathcal{D}(\alpha, \tau) \ni z \rightarrow 0} \frac{g(z) - g(0)}{z} = \lim_{\mathcal{D}(\alpha, \tau) \ni z \rightarrow 0} \frac{g(z) - 1}{z}$$

exists (call it k) and is non-zero. Therefore k is a negative real number. Thus, there exist constants $\epsilon > 0$ and $\kappa \geq 1$ such that $t \in \mathcal{D}(\alpha, \tau)$ whenever $0 < t \leq \epsilon$ and

$$(4.12) \quad 0 \leq 1 - \kappa t \leq g(t) \leq 1 - \kappa^{-1}t \quad \forall t : 0 < t \leq \epsilon.$$

Then for t_1, t_2 such that $0 < t_1 < t_2 \leq \epsilon$, we have

$$\begin{aligned} k_{\mathcal{D}(\alpha, \tau)}(t_1, t_2) &= k_{\mathbb{D}}(g(t_1), g(t_2)) = \frac{1}{2} \log \left(\frac{(1 + g(t_1))(1 - g(t_2))}{(1 + g(t_2))(1 - g(t_1))} \right) \\ &\leq \frac{1}{2} \log 2 + \frac{1}{2} \log \left(\frac{1 - g(t_2)}{1 - g(t_1)} \right) \\ &\leq \frac{1}{2} \log 2 + \log(\kappa) + \frac{1}{2} \log(t_2/t_1), \end{aligned}$$

by (4.12). So for $\xi \in \partial\Omega$ and $t, s \in [0, \infty)$ arbitrary,

$$\begin{aligned} k_{\Omega}(\sigma_{\xi}(t), \sigma_{\xi}(s)) &= k_{\Omega}(\Psi_{\xi}(\epsilon e^{-2t}), \Psi_{\xi}(\epsilon e^{-2s})) \\ &\leq k_{\mathcal{D}(\alpha, \tau)}(\epsilon e^{-2t}, \epsilon e^{-2s}) \\ &\leq \log(\sqrt{2}\kappa) + (1/2) \log(\epsilon e^{-2t}/\epsilon e^{-2s}) \\ &= \log(\sqrt{2}\kappa) + (s - t), \end{aligned}$$

provided $s \geq t$, where Ψ_{ξ} is as introduced in Proposition 4.2. In general

$$(4.13) \quad k_{\Omega}(\sigma_{\xi}(t), \sigma_{\xi}(s)) \leq \log(\sqrt{2}\kappa) + |s - t| \quad \forall s, t \in [0, \infty).$$

Consider the complex affine hyperplane $\xi + T_{\xi}^{\mathbb{C}}(\partial\Omega)$ tangent to $\partial\Omega$ at ξ . Of course, $\xi + T_{\xi}^{\mathbb{C}}(\partial\Omega)$ is a complex affine supporting hyperplane for Ω at ξ . For $t \in \mathbb{R}$ arbitrary, the distance of $\sigma_{\xi}(t)$ from $\xi + T_{\xi}^{\mathbb{C}}(\partial\Omega)$ is clearly ϵe^{-2t} . Consequently, by Result 3.1,

$$(4.14) \quad k_{\Omega}(\sigma_{\xi}(t), \sigma_{\xi}(s)) \geq (1/2) |\log(\epsilon e^{-2t}/\epsilon e^{-2s})| = |s - t| \quad \forall s, t \in [0, \infty).$$

By (4.13) and (4.14), each σ_{ξ} is a $(1, \log(\sqrt{2}\kappa))$ -quasi-geodesic. It only remains to prove the Lipschitz nature of σ_{ξ} .

By the fact that the boundary of Ω is \mathcal{C}^1 , we can, by shrinking ϵ if necessary, ensure that for every $\xi \in \partial\Omega$, $\xi + \mathcal{B}_{\epsilon}\eta^{\xi} \subset \Omega$, where

$$\mathcal{B}_{\epsilon} := \{\zeta \in \mathbb{C} \mid 0 < \operatorname{Re}(\zeta) < 2\epsilon, |\operatorname{Im}(\zeta)| < \operatorname{Re}(\zeta)\}.$$

Elementary two-dimensional geometry then shows that there is a $C > 0$ such that for every $\xi \in \partial\Omega$ and every $t \in [0, \infty)$,

$$r_{\Omega}(\sigma_{\xi}(t), \sigma'_{\xi}(t)) \geq C\epsilon e^{-2t}.$$

(In fact, given that $\xi + \mathcal{B}_{\epsilon}\eta^{\xi} \subset \Omega$, $C = 1/\sqrt{2}$ would work.) Therefore, by Graham's estimate, i.e., Result 3.3, for every $\xi \in \partial\Omega$ and every $t \in [0, \infty)$,

$$\kappa_{\Omega}(\sigma_{\xi}(t), \sigma'_{\xi}(t)) \leq \frac{\|\sigma'_{\xi}(t)\|}{r_{\Omega}(\sigma_{\xi}(t), \sigma'_{\xi}(t))} \leq \frac{2\epsilon e^{-2t}}{C\epsilon e^{-2t}} = \frac{2}{C}.$$

Consequently, for every $\xi \in \partial\Omega$ and every $s, t \in [0, \infty)$,

$$(4.15) \quad k_{\Omega}(\sigma_{\xi}(s), \sigma_{\xi}(t)) \leq (2/C)|s - t|.$$

Therefore, by (4.13), (4.14), and (4.15), it follows that for every $\xi \in \partial\Omega$, σ_{ξ} is a K -almost-geodesic, where $K := \max\{\sqrt{2}\kappa, 2/C\}$. \square

An outcome of one half of our argument for Proposition 4.5 is the following:

Proposition 4.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain having the properties common to Ω_1 and Ω_2 as in the statement of Theorem 1.3. Let $z_0 \in \Omega$. Then, there exists a constant $C > 0$ such that*

$$k_{\Omega}(z_0, z) \leq C + \frac{1}{2} \log \left(\frac{1}{\text{dist}(z, \Omega^c)} \right) \quad \forall z \in \Omega.$$

Remark 4.7. Since the domain Ω in the statement of Proposition 1.5 is bounded, whence $\partial\Omega$ is compact, it is easy to see that Proposition 1.5 is a special case of the above.

Proof. We abbreviate $\text{dist}(z, \Omega^c)$ to $\delta_{\Omega}(z)$. From the argument leading up to (4.13) in the proof of Proposition 4.5, we conclude that there exist constants $\epsilon > 0$ and $K \geq 1$ such that for every $\xi \in \partial\Omega$ and σ_{ξ} as in that proposition,

$$(4.16) \quad k_{\Omega}(\sigma_{\xi}(t), \sigma_{\xi}(s)) \leq |s - t| + \log K \quad \forall s, t \in [0, \infty).$$

By compactness, there exists a $\delta \in (0, \epsilon)$ such that $\text{dist}(\{\sigma_{\xi}(0) \mid \xi \in \partial\Omega\}, \Omega^c) \geq \delta$. It suffices to show that there exists $C > 0$ such that for every $z \in \Omega$ with $\delta_{\Omega}(z) < \delta$, $k_{\Omega}(z_0, z) \leq C + 2^{-1} \log(1/\delta_{\Omega}(z))$. Let $C' := \sup_{\xi \in \partial\Omega} k_{\Omega}(z_0, \sigma_{\xi}(0))$. So let $z \in \Omega$ with $\delta_{\Omega}(z) < \delta$. Now fix a $\xi \in \partial\Omega$ such that $\|z - \xi\| = \delta_{\Omega}(z)$. Clearly, then, there exists $t(z) \in (0, \infty)$ such that $z = \sigma_{\xi}(t(z))$. So $k_{\Omega}(z_0, z) \leq k_{\Omega}(z_0, \sigma_{\xi}(0)) + k_{\Omega}(\sigma_{\xi}(0), \sigma_{\xi}(t(z))) \leq C' + t(z) + \log K$, by (4.16). Since, by definition of σ_{ξ} , $\delta_{\Omega}(z) = \epsilon e^{-2t(z)}$, a simple calculation shows that $t(z) \leq 2^{-1} \log(1/\delta_{\Omega}(z))$. Therefore $k_{\Omega}(z_0, z) \leq C + 2^{-1} \log(1/\delta_{\Omega}(z))$, where $C := C' + \log K$. This gives the desired conclusion. \square

5. THE PROOF OF THEOREM 1.3

The proof of Theorem 1.3 requires the following conclusions: if $\Omega \subset \mathbb{C}^n$ is a domain that has the properties possessed in common by Ω_1 and Ω_2 in the statement of Theorem 1.3, then (we remind the reader that for $\xi \in \partial\Omega$, the set $(\xi + T_{\xi}^{\mathbb{C}}(\partial\Omega))$ is as described in Section 3):

- (1) If $\xi \in \partial\Omega$ and $(p_{\nu})_{\nu \geq 1}, (q_{\mu})_{\mu \geq 1}$ are sequences in Ω converging to ξ , then

$$\lim_{\nu, \mu \rightarrow \infty} (p_{\nu} | q_{\mu})_o = \infty.$$

- (2) If $\xi, \xi' \in \partial\Omega$ and $(p_{\nu})_{\nu \geq 1}, (q_{\mu})_{\mu \geq 1}$ are sequences in Ω converging to ξ and ξ' , respectively, such that

$$\limsup_{\nu, \mu \rightarrow \infty} (p_{\nu} | q_{\mu})_o = \infty,$$

$$\text{then } \xi + T_{\xi}^{\mathbb{C}}(\partial\Omega) = \xi' + T_{\xi'}^{\mathbb{C}}(\partial\Omega).$$

In the above, $(\cdot | \cdot)_o$ denotes the Gromov product relative to the Kobayashi distance on Ω and with respect to an arbitrary but fixed base point $o \in \Omega$. It is defined as

$$(x | y)_o := \frac{1}{2} (k_{\Omega}(x, o) + k_{\Omega}(y, o) - k_{\Omega}(x, y)).$$

The above conclusions have been demonstrated by Zimmer under the conditions he states in [15, Theorem 4.1]. We observe that what has actually been established

in [15, Theorem 4.1] is the following:

Proposition 5.1. *Suppose Ω is a bounded open convex subset of \mathbb{C}^n having \mathcal{C}^1 -smooth boundary. Suppose Ω possesses the property that there exist constants $\epsilon > 0, K \geq 1$ such that, for each $\xi \in \partial\Omega$, the path*

$$\sigma_\xi := t \mapsto \xi + \epsilon e^{-2t} \eta^\xi : [0, \infty) \rightarrow \Omega$$

is a K -almost-geodesic. Then:

- (1) *If $\xi \in \partial\Omega$ and $(p_\nu)_{\nu \geq 1}, (q_\mu)_{\mu \geq 1}$ are sequences in Ω converging to ξ , then*

$$\lim_{\nu, \mu \rightarrow \infty} (p_\nu | q_\mu)_o = \infty.$$

- (2) *If $\xi, \xi' \in \partial\Omega$ and $(p_\nu)_{\nu \geq 1}, (q_\mu)_{\mu \geq 1}$ are sequences in Ω converging to ξ and ξ' , respectively, such that*

$$\limsup_{\nu, \mu \rightarrow \infty} (p_\nu | q_\mu)_o = \infty,$$

$$\text{then } \xi + T_\xi^{\mathbb{C}}(\partial\Omega) = \xi' + T_{\xi'}^{\mathbb{C}}(\partial\Omega).$$

The condition on $\partial\Omega$ in [15, Theorem 4.1] was required to obtain the property concerning the paths $\{\sigma_\xi \mid \xi \in \partial\Omega\}$ stated in Proposition 5.1. Other than this, there is absolutely no difference between the proofs of [15, Theorem 4.1] and Proposition 5.1. We therefore omit the proof of the latter.

Finally, we give the proof of Theorem 1.3.

Proof. First, we show that whenever $\xi \in \partial\Omega_1$, $\lim_{z \rightarrow \xi} F(z)$ exists. Since F is an isometry with respect to the Kobayashi distances, we see, from the definition of the Gromov product above, that, for every $z, w, o \in \Omega_1$,

$$(z|w)_o = (F(z)|F(w))_{F(o)}.$$

First note that if $\xi \in \partial\Omega_1$ and $(z_\nu)_{\nu \geq 1}$ is a sequence in Ω_1 converging to ξ such that $(F(z_\nu))_{\nu \geq 1}$ converges to some point $\zeta \in \overline{\Omega}_2$, then $\zeta \in \partial\Omega_2$. The reason is that, if we fix a point $o \in \Omega_1$ arbitrarily, we see that

$$\lim_{\nu \rightarrow \infty} k_{\Omega_2}(F(z_\nu), F(o)) = \lim_{\nu \rightarrow \infty} k_{\Omega_1}(z_\nu, o) = \infty,$$

by Lemma 3.1. Consequently, $\zeta = \lim_{\nu \rightarrow \infty} F(z_\nu)$ must belong to $\partial\Omega_2$. Thus, if $\xi \in \partial\Omega_1$ and $(z_\nu)_{\nu \geq 1}, (w_\nu)_{\nu \geq 1}$ are sequences in Ω_1 converging to ξ such that $(F(z_\nu))_{\nu \geq 1}$ and $(F(w_\nu))_{\nu \geq 1}$ converge to $\zeta_1, \zeta_2 \in \overline{\Omega}_2$, respectively, then $\zeta_1, \zeta_2 \in \partial\Omega_2$. Moreover,

$$\lim_{\nu \rightarrow \infty} (F(z_\nu)|F(w_\nu))_{F(o)} = \lim_{\nu \rightarrow \infty} (z_\nu|w_\nu)_o = \infty,$$

by (1) of Proposition 5.1 above. Consequently, by (2) of the same proposition, $\zeta_1 + T_{\zeta_1}^{\mathbb{C}}(\partial\Omega_2) = \zeta_2 + T_{\zeta_2}^{\mathbb{C}}(\partial\Omega_2)$. Therefore, since Ω_2 is \mathbb{C} -strictly convex, one has $\zeta_1 = \zeta_2$. Since Ω_2 is bounded so that any sequence in it has a convergent subsequence, the above shows that $\lim_{z \rightarrow \xi} F(z)$ exists.

Then define $\tilde{F} : \overline{\Omega}_1 \rightarrow \overline{\Omega}_2$ by letting \tilde{F} equal F on Ω_1 and by letting $\tilde{F}(\xi)$, for $\xi \in \partial\Omega_1$, be $\lim_{z \rightarrow \xi} F(z)$. It is routine to show that \tilde{F} is continuous, in view of the conclusions of the previous paragraph. This completes the proof. \square

ACKNOWLEDGMENT

The author thanks the referee of this paper for helpful suggestions concerning the exposition in this work.

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