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Notions of visibility with respect to the Kobayashi distance: comparison and applications

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Abstract
In this article, we study notions of visibility with respect to the Kobayashi distance for relatively compact complex submanifolds in Euclidean spaces. We present a sufficient condition for a domain to possess the visibility property relative to Kobayashi almost-geodesics introduced by Bharali–Zimmer (we call this simply *the visibility property*). As an application, we produce new classes of domains having this kind of visibility. Next, we introduce and study the notion of *visibility subspaces* of relatively compact complex submanifolds. Using this notion, we generalize to such submanifolds a recent result of Bracci–Nikolov–Thomas. The utility of this generalization is demonstrated by proving a theorem on the continuous extension of Kobayashi isometries. Finally, we prove a Wolff–Denjoy-type theorem that is a generalization of recent results of this kind by Bharali–Zimmer and Bharali–Maitra and that, owing to the new classes of domains mentioned, is a proper generalization. Along the way, we note that what is needed for the proof of this sort of theorem to work is a form of visibility that seems to be intermediate between what we are calling *visibility* and visibility with respect to ordinary Kobayashi geodesics.

Keywords Kobayashi distance · Kobayashi metric · Embedded submanifolds · Taut submanifolds · Kobayashi isometry · Visibility · Wolff–Denjoy theorem

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1 Introduction and statement of main results

For a hyperbolic complex manifold M, the Kobayashi distance k_M and the Kobayashi–Royden pseudometric κ_M encapsulate many complex-geometric and function-theoretic properties of M. Recall that k_M is the integrated form of κ_M (see Result 2.1). This article explores a particular aspect of the Kobayashi distance, namely, notions of visibility with respect to real geodesics and almost-geodesics with respect to the Kobayashi distance (see Definition 2.5 in Sect. 2). The two notions of visibility that we shall focus on originated in the articles [2–4, 19], where these properties were studied for bounded domains in complex Euclidean spaces. The motivation for the above two notions is to capture the negative-curvature-type behaviour of the Kobayashi distance; see the introductions in [2, 3] for a detailed discussion regarding this. In this article, we shall focus on the analogous properties on bounded, connected, embedded complex submanifolds of $\mathbb{C}^d$. Not only is this interesting in its own right, but we shall see that it leads to some useful applications.

In what follows, M will always denote a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$. We shall use ∂M to denote $\overline{M} \setminus M$, the boundary of M calculated with respect to $\overline{M}$, which is a compact subset of $\mathbb{C}^d$. For $z \in M$, we shall use $\delta_M(z)$ to denote the Euclidean distance between z and ∂M. We now introduce the aforementioned notions of visibility for M with respect to the Kobayashi distance k_M.

Definition 1.1 Let M be as above. Fix $\lambda \geq 1, \kappa > 0$. We say that M *has the visibility property with respect to* (λ, κ)*-almost-geodesics* or that M *is a* (λ, κ)*-visibility submanifold* if the following two properties hold true.

- Any two distinct points of M can be joined by a (λ, κ)-almost-geodesic.
- For every pair of points $p \neq q \in \partial M$, there exist $\mathbb{C}^d$-neighbourhoods V and W of p and q, respectively, and a compact subset K of M such that $\overline{V} \cap \overline{W} = \emptyset$ and such that every (λ, κ)-almost-geodesic in M with initial point in V and terminal point in W intersects K.

If M is a *domain* $\Omega \subset \mathbb{C}^d$ that has the visibility property with respect to (λ, κ)-almost-geodesics, then we say that M *is a* (λ, κ)*-visibility domain*. If M, as above, has the visibility property with respect to (λ, κ)-almost-geodesics for *every* $\lambda \geq 1$ and $\kappa > 0$, then we shall say that M *is a visibility submanifold*. If M has the visibility property with respect to $(1, \kappa)$-almost-geodesics for all $\kappa > 0$, then we shall say that M *is a weak visibility submanifold*. Finally, we shall say that M *is a geodesic visibility submanifold* if (M, k_M) is a complete distance space and M satisfies the second property above in the definition of (λ, κ)-visibility submanifolds with "(λ, κ)-almost-geodesic" replaced by "real Kobayashi geodesic".

Given M as above, a $\lambda \geq 1$ and a $\kappa > 0$, it is not clear whether, given two distinct points in M, there is a (λ, κ)-almost-geodesic joining them. When $M = \Omega$, a bounded domain in $\mathbb{C}^d$, this was proved by Bharali–Zimmer [3, Proposition 4.4]. That this is the case for a general M, as above, is the content of Theorem 2.8 in Sect. 2. In the case where (M, k_M) is complete, any two points in M can be joined by a real geodesic (see Remark 2.6). Therefore, the definitions above are not vacuous. We also mention that for bounded *domains* the concept of visibility has been studied in the articles [2, 3]. The concept of geodesic visibility in the context of (bounded) domains Ω for which $(\Omega, \mathsf{k}_\Omega)$ is complete has been studied in the recent article [4].

We now turn our attention to visibility domains. In [3], it was shown that a large class of domains, called *Goldilocks domains*, possesses the visibility property. Since we shall refer to them several times in this work, let us introduce them here. For this, we give the following definition. Given M as above and given an open set $U \subset \mathbb{C}^d$ such that $U \cap \partial M \neq \emptyset$, we define, for all $r > 0$,

$$\mathfrak{M}_{M,\,U}(r) := \sup\left\{\frac{1}{\kappa_M(z;\,v)} \mid z \in U \cap M : \delta_M(z) \leq r \text{ and } v \in T_z^{(1,0)}M : \|v\| = 1\right\}, \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm and $T_z^{(1,0)}M$ denotes the complex tangent space to M at z. We abbreviate $\mathfrak{M}_{M,\,\mathbb{C}^d}$ to $\mathfrak{M}_M$. Note that, in particular, we can take M to be a bounded domain in $\mathbb{C}^d$.

Definition 1.2 A bounded domain $\Omega \subset \mathbb{C}^d$ is called a *Goldilocks domain* if

(1) for some (hence any) $\epsilon > 0$ we have

$$\int_0^{\epsilon} \frac{1}{r}\mathfrak{M}_\Omega(r)\,dr < \infty, \text{ and}$$

(2) for each $z_0 \in \Omega$, there exist constants $C, \alpha > 0$ (that depend on z_0) such that

$$\mathsf{k}_\Omega(z_0, z) \leq C + \alpha \log \frac{1}{\delta_\Omega(z)} \quad \forall z \in \Omega.$$

Examples of Goldilocks domains are given in Sect. 2 of [3]. In particular, due to a result of S. Cho [7], every bounded smooth pseudoconvex domain of finite D'Angelo type is a Goldilocks domain. Later, Bharali–Maitra in [2] gave another criterion more permissive than the one in Definition 1.2 for domains to possess the visibility property. Using this new criterion, they also constructed domains, which they called *caltrops*, that possess the visibility property but are not Goldilocks domains (more precisely, they do not satisfy condition (2) in Definition 1.2). Based on the **new** understanding that the proof that a domain possesses the visibility property *can be localized to the boundary*, we present a sufficient condition more permissive than the one given in [2, Theorem 1.5] for a domain to possess the visibility property. (Some time after this paper was announced, results were published (see [13, 16]) that have substantiated the assertion that visibility itself is a local property of the boundary.)

Theorem 1.3 (Extended visibility lemma) *Let Ω be a bounded domain in $\mathbb{C}^d$. Let $E \subset \partial\Omega$ be a closed set such that for every $p \neq q \in \partial\Omega$, there exist $p' \in \partial\Omega$ and $r > 0$ satisfying*

(a) *$p \in B(p', r)$ and $q \in \partial\Omega \setminus \overline{B(p', r)}$;*
(b) *$E \cap \partial B(p', r) = \emptyset$.*

Further, assume that for every $q' \in \partial\Omega \setminus E$ there exist a neighbourhood U of q', a point $z_0 \in \Omega$ and a $\mathcal{C}^1$-smooth strictly increasing function $f : (0, \infty) \longrightarrow \mathbb{R}$, with $f(t) \to \infty$ as $t \to \infty$, such that

(1) *for all $z \in \Omega \cap U$, $\mathsf{k}_\Omega(z_0, z) \leq f(1/\delta_\Omega(z))$;*
(2) *$\mathfrak{M}_{\Omega,\,U}(r) \to 0$ as $r \to 0$, and*
(3) *there exists $r_0 > 0$ such that*

$$\int_0^{r_0} \frac{\mathfrak{M}_{\Omega,\,U}(r)}{r^2} f'\left(\frac{1}{r}\right) dr < \infty.$$

Then Ω is a visibility domain.

Here, $B(p', r)$ denotes the open Euclidean ball of radius r centred at p' in $\mathbb{C}^d$.

As an application of the above theorem, we prove the following corollary.

Corollary 1.4 *Let Ω be a bounded domain in $\mathbb{C}^d$. Suppose there exists a compact subset $E \subset \partial\Omega$ such that E_a, the set of limit points of E, is a finite set and such that every $p \in \partial\Omega \setminus E$ is a smooth pseudoconvex boundary point of finite D'Angelo type. Then Ω is a visibility domain.*

Remark 1.5 A recent result of Bracci–Nikolov–Thomas says that every bounded convex domain with $\mathcal{C}^\infty$-boundary is a geodesic visibility domain if all except finitely many boundary points of the domain are of finite D'Angelo type; see [4, Theorem 1.1]. This result can be deduced from Corollary 1.4. To see this, observe that, by the above corollary, any such domain is a visibility domain, hence a weak visibility domain. Since bounded convex domains are complete, it follows from Corollary 3.2 (see Sect. 3.2) that any such domain is a geodesic visibility domain.

Remark 1.6 Some time after this paper was announced, Bharali–Zimmer generalized the above theorem (see [1, Theorem 1.4]) by removing the assumption of the boundedness of the domain and by allowing E to be any closed, totally disconnected subset of the boundary.

We turn our attention to geodesic visibility. We first remark that it is easy to come up with examples where the complete distance space (M, k_M) does not have geodesic visibility although there are subspaces of M that may have this property. This possibility motivated us to introduce the following definitions. Before giving them, we clarify that, given a curve γ, we will use the symbol γ itself to denote $\mathsf{ran}(\gamma)$. However, if there is any danger of confusion, we will use the unambiguous notation $\mathsf{ran}(\gamma)$. We now give

Definition 1.7 Let M be a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$. A subset S of M will be called a *geodesic subspace* if the following two conditions are satisfied.

- The distance space $(S, \mathsf{k}_M|_{S\times S})$ is Cauchy-complete.
- For any two distinct points in S, there exists a geodesic of the space (M, k_M) that passes through those points and that is contained in S.

Definition 1.8 A geodesic subspace S is called a *visibility subspace* of M if for any $p \neq q \in \partial_a S := \overline{S} \setminus S$, there exist $\mathbb{C}^d$-neighbourhoods U and V of p and q, respectively, and a compact subset K of S such that $\overline{U} \cap \overline{V} = \emptyset$ and such that, for every k_M-geodesic γ in S with initial point in $U \cap S$ and terminal point in $V \cap S$, $\mathsf{ran}(\gamma) \cap K \neq \emptyset$.

In the case $S = M$, S being a visibility subspace is equivalent to S being a geodesic visibility submanifold. Denoting the open unit disk in $\mathbb{C}$ by $\mathbb{D}$, we ask the reader to note that $(\mathbb{D}^n, \mathsf{k}_{\mathbb{D}^n})$, $n \geq 2$, is not a geodesic visibility domain (this is easy to see). It is known (see [9]) that all the one-dimensional retracts of $\mathbb{D}^n$, $n \geq 2$, are given by $V_f := \{(z, f(z)) : z \in \mathbb{D}\}$, where $f = (f_1, \ldots, f_{n-1}) : \mathbb{D} \longrightarrow \mathbb{D}^{n-1}$ is a holomorphic map. It is not too difficult to show that V_f is a visibility subspace of $\mathbb{D}^n$ if each f_j extends continuously to $\overline{\mathbb{D}}$. We sketch an argument showing this at the beginning of Sect. 5. The converse is also true, namely, if V_f is a visibility subspace of $\mathbb{D}^n$, each f_j extends continuously to $\overline{\mathbb{D}}$. One can see this by noting that V_f is actually the image of a complex geodesic in $\mathbb{D}^n$, namely of $z \mapsto (z, f(z)) : \mathbb{D} \longrightarrow \mathbb{D}^n$, and then using Theorem 1.10 below.

We now present a result regarding the visibility of geodesic subspaces. This result is a generalization of Theorem 3.3 in [4] to the context of geodesic subspaces. This theorem might seem overly abstract, but its utility will become apparent when we prove Theorem 1.10 and its corollaries.

Theorem 1.9 *Let M be a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$. Let $S \subset M$ be a geodesic subspace of M such that $(S, \mathsf{k}_M|_{S\times S})$ is Gromov hyperbolic. Then S is a visibility subspace of M if and only if the identity map $\mathsf{id}_S : S \longrightarrow S$ extends to a continuous surjective map $\widehat{\mathsf{id}}_S : \overline{S}^G \longrightarrow \overline{S}$, where $\overline{S}^G$ denotes the Gromov compactification of S with respect to $\mathsf{k}_M|_{S\times S}$. Moreover, this extended map is a homeomorphism if and only if S has no geodesic loops in $\overline{S}$.*

We refer the reader to Bridson and Haefliger [5, Part III, Chapter 3] for the definition of the Gromov compactification of a proper, geodesic distance space that is Gromov hyperbolic. (Also see [4, Sect. 3] for a quick introduction to the same when the distance is the Kobayashi distance.) We also refer the reader to Definition 5.2 in Sect. 5 for the definition of a geodesic loop.

Using the above theorem, we prove a result concerning the continuous extension of Kobayashi isometries.

Theorem 1.10 *Suppose that $M \subset \mathbb{C}^m$ and $N \subset \mathbb{C}^n$ are bounded, connected, embedded complex submanifolds and that (M, k_M) is a complete Gromov hyperbolic space. Let $f : M \longrightarrow N$ be an isometric embedding with respect to the Kobayashi distances and suppose that $S := f(M)$ (which is easily seen to be a geodesic subspace of N) is a visibility subspace of N. Then f extends to a continuous map $\widehat{f} : \overline{M}^G \longrightarrow \overline{N}$, where $\overline{M}^G$ denotes the Gromov compactification of (M, k_M). Further, if S has no geodesic loops in $\overline{S}$, then $\widehat{f}$ is a homeomorphism from $\overline{M}^G$ to $\overline{S}$.*

To illustrate the use of this theorem, we provide the following corollary, which partly generalizes [12, Theorem 1.3]. But first a few words about two concepts that occur in the statement of the corollary below. The first one is that of the $\mathcal{C}^{1,\mathrm{Dini}}$-smoothness of the boundary of a domain in $\mathbb{C}^d$. Since we are not going to make explicit use of $\mathcal{C}^{1,\mathrm{Dini}}$-smoothness, we shall not define it here. We direct the reader to Nikolov and Andreev [14] for the definition. We note only that if a domain has $\mathcal{C}^{1,\alpha}$-smooth boundary, where $\alpha > 0$ is arbitrary, then it automatically has $\mathcal{C}^{1,\mathrm{Dini}}$-smooth boundary. In particular, all domains with $\mathcal{C}^k$-smooth boundary, where $k \geq 2$, have $\mathcal{C}^{1,\mathrm{Dini}}$-smooth boundary. The second concept is that of $\mathbb{C}$-strict convexity, which we shall define in detail. Before giving the definition, we recall that, given a domain $\Omega \subset \mathbb{C}^d$ and a $\mathcal{C}^1$-smooth boundary point p of Ω, we can consider the (real) tangent space $T_p(\partial\Omega)$, where we view it *extrinsically* (i.e., as a real hyperplane in $\mathbb{C}^d$), and we can also consider *the complex tangent space to $\partial\Omega$ at p*, given by $H_p(\partial\Omega) := T_p(\partial\Omega) \cap i\big(T_p(\partial\Omega)\big)$.

Definition 1.11 Given a convex domain $\Omega \subset \mathbb{C}^d$ with $\mathcal{C}^1$-smooth boundary, a boundary point p of Ω is said to be *a $\mathbb{C}$-strictly convex boundary point* if $\big(p + H_p(\partial\Omega)\big) \cap \overline{\Omega} = \{p\}$, where $H_p(\partial\Omega)$ is the complex tangent space to $\partial\Omega$ at p.

Corollary 1.12 *Let Ω be a bounded convex domain in $\mathbb{C}^d$. Suppose that there exists a compact subset $S \subset \partial\Omega$ such that S_a, the set of limit points of S, is a finite set, and such that every $p \in \partial\Omega \setminus S$ is a $\mathcal{C}^{1,\mathrm{Dini}}$-smooth $\mathbb{C}$-strictly convex boundary point. Let $f : \mathbb{D} \longrightarrow \Omega$ be a complex geodesic. Then f extends continuously to $\overline{\mathbb{D}}$.*

The proof of the above corollary is given at the end of Sect. 5.

Finally, we move to a topic in the theory of iterations of a holomorphic self-map, where the visibility property turns out to imply some interesting consequences. In this direction, we begin with the following famous result due, independently, to Denjoy and Wolff [8, 18].

Result 1.13 *(Denjoy, Wolff) Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic map. Either f has a unique fixed point in $\mathbb{D}$ or there exists a point $p \in \partial\mathbb{D}$ such that $f^n(z) \to p$ as $n \to \infty$ for each $z \in \mathbb{D}$. In the latter case, convergence is uniform on compact subsets of $\mathbb{D}$.*

In several works, see e.g. [2, 3, 19], it is noted that similar phenomena regarding the iteration of a holomorphic self-map of a bounded domain—as exhibited above in the case of $\mathbb{D}$—could be understood and explained by appealing to the visibility property of the Kobayashi distance. Using this connection, Bharali–Maitra proved two Wolff–Denjoy-type theorems [2, Theorems 1.8 and 1.9] for taut domains possessing the visibility property. Our next result improves upon [2, Theorem 1.8] in two ways: our result is a Wolff–Denjoy-type theorem for bounded, taut *submanifolds* of $\mathbb{C}^d$ on which only **weak** visibility is assumed. We now present the result.

Theorem 1.14 *Suppose that M is a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$ that satisfies the weak visibility property and is taut. Let $F : M \longrightarrow M$ be a holomorphic map. Then exactly one of the following holds:*

(1) *For every $z \in M$, $\{F^\nu(z) : \nu \in \mathbb{Z}_+\}$ is a relatively compact subset of M;*
(2) *there exists $\xi \in \partial M$ such that, for every $z \in M$, $\lim_{\nu\to\infty} F^\nu(z) = \xi$, this convergence being uniform on the compact subsets of M.*

Remark 1.15 The method of our proof follows very closely that of [2, Theorem 1.8]. A crucial tool that was employed in the latter proof (see [2, Theorem 4.3]) was a consequence of the fact that $\lim_{r\to 0} \mathfrak{M}_\Omega(r) = 0$ for any taut visibility domain Ω. We prove an analogous result (Theorem 6.4 in Sect. 6) for an M as above that is taut and that is only assumed to have the *weak* visibility property. We emphasize that our theorem is particularly useful where the given submanifold is not known to possess the visibility property, but *is* known to possess the *weak* visibility property; see e.g. Example 4.1 in Sect. 4.

We now present the plan of this paper. In Sect. 2, we present preliminary material relating to the Kobayashi distance and metric on relatively compact complex submanifolds of $\mathbb{C}^d$ and prove that almost-geodesics joining arbitrary pairs of points exist on any such manifold. In Sect. 3, we present the proof of Theorem 1.3 and then compare the various notions of visibility that appear in this paper. In Sect. 4, we present two examples of domains that are not Goldilocks domains but nevertheless possess some form of visibility. In Sect. 5, we study basic properties of geodesic and visibility subspaces and then prove Theorems 1.9 and 1.10. Finally, in Sect. 6, we prove Theorem 1.14 after proving some preliminary results, which are interesting in their own right.

2 Preliminaries

In this section, we shall show the existence of (λ, κ)-almost-geodesics (with respect to the Kobayashi distance) for a bounded, connected, embedded complex submanifold M of $\mathbb{C}^d$. Before we begin, we recall that the Kobayashi pseudometric κ_M is upper semicontinuous. Therefore, for any continuous mapping $\gamma : I \longrightarrow T^{(1,0)}M$, where $I \subset \mathbb{R}$ is an interval and $T^{(1,0)}M$ denotes the complex tangent bundle of M,

$$\int_I \kappa_M(\gamma(t))dt$$

makes sense (it may be ∞). Therefore, if we have an embedded complex submanifold M of $\mathbb{C}^d$ and we have a piecewise $\mathcal{C}^1$ curve $\gamma : [a,b] \longrightarrow M$, where $a, b \in \mathbb{R}$, $a \neq b$, then

$$\int_a^b \kappa_M(\gamma(t), \gamma'(t))dt < \infty. \tag{2.1}$$

It also follows easily that if $\gamma : [a,b] \longrightarrow M$ is an absolutely continuous curve, then the Lebesgue integral of the function $t \mapsto \kappa_M(\gamma(t), \gamma'(t)) : [a,b] \longrightarrow [0,\infty)$ is defined (a priori, it could be ∞). Therefore, given $\gamma : [a,b] \longrightarrow M$, an absolutely continuous curve, we let $l_M(\gamma)$ denote the integral (2.1), *the length of γ calculated using the Kobayashi pseudometric*. In what follows, the following result is relevant.

Result 2.1 *Let M be a connected, embedded complex submanifold of $\mathbb{C}^d$.*

(1) *[15, Theorem 1] For any $z, w \in M$, we have*

$$\mathsf{k}_M(z,w) = \inf \left\{ l_M(\gamma) \mid \gamma : [a,b] \longrightarrow M \text{ is piecewise } \mathcal{C}^1, \right.$$
$$\left. \text{with } \gamma(a) = z \text{ and } \gamma(b) = w \right\}.$$

We can also take γ to be $\mathcal{C}^1$ above.

(2) *[17, Theorem 3.1] For any $z, w \in M$, we have*

$$\mathsf{k}_M(z,w) = \inf \left\{ l_M(\gamma) \mid \gamma : [a,b] \longrightarrow M \text{ is absolutely continuous, with } \gamma(a) = z \text{ and } \gamma(b) = w \right\}.$$

We now present a result that is at the heart of the main result of this section, namely the existence of (λ, κ)-almost-geodesics.

Proposition 2.2 *Let M be a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$. Then the following hold.*

(1) *There exists $c > 0$ such that*

$$c\,\|X\| \leq \kappa_M(z,X)$$

for all $z \in M$ and $X \in T_z^{(1,0)}M$.

(2) *For any compact set $K \subset M$, there exists a constant $C_1 = C_1(K) > 0$ so that*

$$\kappa_M(z,X) \leq C_1\,\|X\|$$

for all $z \in K$ and $X \in T_z^{(1,0)}M$.

Remark 2.3 Part (1) together with Result 2.1 implies immediately that $c\,\|z-w\| \leq \mathsf{k}_M(z,w)$ for all $z, w \in M$. Similarly, working with Part (2) above, one can show that for any compact set $K \subset M$, there exists a constant $C_2 = C_2(K) > 0$ such that $\mathsf{k}_M(z,w) \leq C_2\,\|z-w\|$ for all $z, w \in K$.

Before we prove the proposition above, we shall state a result that will be used in the proof of the proposition.

Result 2.4 [15, Proposition 2] *Let $\mathscr{M}$ be a complex manifold of dimension n. If a compact set $K \subset \mathscr{M}$ is contained in a coordinate polydisk, then there exists a constant $C = C(K)$ such that*

$$\kappa_{\mathscr{M}}(z,X) \leq C\|X\|$$

for all $z \in K$, $X \in T_z^{(1,0)}\mathscr{M}$.

A co-ordinate polydisk in $\mathscr{M}$ is essentially a co-ordinate chart $(\psi, U, \psi(U))$, $U \subset \mathscr{M}$ being open, such that $\psi(U)$ is a polydisk in $\mathbb{C}^n$.

Proof of Proposition 2.2 The proof of part (1) of the proposition is closely analogous to that of part (1) of Bharali and Zimmer [3, Proposition 3.5], so we omit it here.

To establish part (2), we choose, for each $z \in K$, a coordinate polydisk U_z of M centred at z. Let $U_z' \subset U_z$ be another coordinate polydisk centred at z that is relatively compact in U_z for all $z \in K$. Since K is compact, there are finitely many elements of $\{U_z' : z \in K\}$ that cover K. Let $\{U_{z_i}'\}_{i=1}^k$ be a finite cover, for some $k \in \mathbb{Z}_+$. Then, since $\overline{U}_{z_i}'$ is a compact subset of U_{z_i} for all $i = 1, \ldots, k$, by Result 2.4,

$$\kappa_M(z, X) \leq C_i \|X\|$$

for all $z \in U_{z_i}'$ and $X \in T_z^{(1,0)}M$, where C_i is a constant depending on the compact set $\overline{U}_{z_i}'$. Set $C = C(K) := \max\{C_i : i = 1, \ldots, k\}$. Then

$$\kappa_M(z, X) \leq C \|X\|$$

for all $z \in K \subset \bigcup U_{z_i}'$ and $X \in T_z^{(1,0)}M$. This shows that part (2) is true. □

Definition 2.5 Let $M \subset \mathbb{C}^d$ be as before and let $I \subset \mathbb{R}$ be an interval. A *real Kobayashi geodesic* is a map $\sigma : I \longrightarrow M$ that is an isometric embedding, i.e., for any $s, t \in I$, we have

$$|s - t| = \mathsf{k}_M(\sigma(s), \sigma(t)).$$

For $\lambda \geq 1$ and $\kappa \geq 0$, a curve $\sigma : I \longrightarrow M$ is called a (λ, κ)-*almost-geodesic* if

(1) for all $s, t \in I$, $(1/\lambda)|t - s| - \kappa \leq \mathsf{k}_M\big(\sigma(t), \sigma(s)\big) \leq \lambda|t - s| + \kappa;$ and
(2) σ is absolutely continuous, so that $\sigma'(t)$ exists for almost every $t \in I$, and, for almost every $t \in I$, $\kappa_M(\sigma(t), \sigma'(t)) \leq \lambda$.

A curve $\sigma : I \longrightarrow M$ that is only required to satisfy condition (1) above is called a (λ, κ)-*quasi-geodesic*. Note that such curves are not necessarily continuous.

Remark 2.6 It is a fact that given M as above, the distance space (M, k_M) is a *length space* that is locally compact. It is a consequence of the Hopf–Rinow Theorem that if (M, k_M) is (Cauchy-) complete, then given any two points $p \neq q \in M$, there is a real geodesic connecting them, i.e., a geodesic $\sigma : [0, \mathsf{k}_M(p, q)] \longrightarrow M$ with $\sigma(0) = p$ and $\sigma(\mathsf{k}_M(p, q)) = q$. It is also a consequence of that theorem that when (M, k_M) is complete, it is proper, i.e., the closed k_M-balls are compact. The reader is referred to Bridson and Haefliger [5, Part I, Chapter 3] for the definition of length space and for the statement of the Hopf–Rinow Theorem.

The following result is the consequence of part (1) of the above proposition. Its proof is exactly the same as in [3, Proposition 4.3]. Therefore we omit it here.

Proposition 2.7 *Let $M \subset \mathbb{C}^d$ be as before. Then for any $\lambda \geq 1$, there exists $C = C(\lambda) > 0$ so that any (λ, κ)-almost-geodesic $\sigma : I \longrightarrow M$ is C-Lipschitz with respect to the Euclidean distance.*

We are now ready to present the principal result of this section.

Theorem 2.8 *Let $M \subset \mathbb{C}^d$ be as before. For any $z, w \in M$ and any $\kappa > 0$, there exists a $(1, \kappa)$-almost-geodesic $\sigma : [a, b] \longrightarrow M$ such that $\sigma(a) = z$ and $\sigma(b) = w$.*

Proof The proof is an adaptation of the proof of [3, Proposition 4.4]. Here we shall only give the main idea of the proof. First, by part (1) of Result 2.1, there exists a $\mathcal{C}^1$ curve $\gamma : [0,\ 1] \longrightarrow M$ such that $\gamma(0) = z$, $\gamma(1) = w$, and such that

$$l_M(\gamma) < \mathsf{k}_M(z, w) + \kappa.$$

In addition, we can assume that $\gamma'(t) \neq 0$ for all $t \in [0,\ 1]$. Now, we consider the arc-length function, namely,

$$f := t \mapsto \int_0^t \kappa_M(\gamma(r), \gamma'(r))dr \ :\ [0, 1] \longrightarrow [0, \infty).$$

Using Proposition 2.2 above, we show that f is a bi-Lipschitz function, and consequently strictly increasing. Let $g : [0, l_M(\gamma)] \longrightarrow [0,\ 1]$ be the inverse of f, that is, $g(f(t)) = t$ for all $t \in [0,\ 1]$. Consider the reparametrization of γ defined by $\sigma := \gamma \circ g$. It is not difficult to show that σ has unit-speed with respect to κ_M. This, in particular, implies that σ is an $(1, \kappa)$-almost-geodesic. We refer the reader to Bharali and Zimmer [3, Proposition 4.4] for more details. □

Note that given $\lambda \geq 1$ and $\kappa > 0$, every $(1, \kappa)$-almost-geodesic is a (λ, κ)-almost-geodesic too. Hence Theorem 2.8 implies the existence of (λ, κ)-almost-geodesics for any $\lambda \geq 1$ and $\kappa > 0$.

We end this section with the following simple result about $(1, \kappa)$-quasi-geodesics that we shall need later in the article. This result is a direct consequence of the definition of $(1, \kappa)$-quasi-geodesic together with the triangle inequality. So we omit the proof.

Result 2.9 *Let $M \subset \mathbb{C}^d$ be as before. If $\sigma : [a, b] \longrightarrow M$ is a $(1, \kappa)$-quasi-geodesic, then for all $t \in [a, b]$ we have*

$$\mathsf{k}_M(\sigma(a), \sigma(b)) \leq \mathsf{k}_M(\sigma(a), \sigma(t)) + \mathsf{k}_M(\sigma(t), \sigma(b)) \leq \mathsf{k}_M(\sigma(a), \sigma(b)) + 3\kappa.$$

3 The Extended Visibility Lemma and relations amongst different types of visibility

In this section, we present the proofs of Theorem 1.3 and Corollary 1.4. As hinted at in the introduction, the proofs of these results demonstrate the fact that the proof that a given domain or submanifold possesses the visibility property is localizable. This realization also motivates us to reconsider geodesic visibility and its relation with weak visibility. This we present in Sect. 3.2. Finally, in Sect. 3.3, we compare visibility and geodesic visibility.

3.1 The proofs of Theorem 1.3 and Corollary 1.4

The proof of Theorem 1.3 Suppose Ω is not a visibility domain. Then there exist $\lambda \geq 1$ and $\kappa > 0$ such that Ω does not have the visibility property with respect to (λ, κ)-almost-geodesics. This implies that there exist $p \neq q \in \partial\Omega$, sequences $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ in Ω that converge to p and q respectively, and a sequence $(\gamma_n)_{n\geq 1}$ of (λ, κ)-almost-geodesics—$\gamma_n : [a_n, b_n] \longrightarrow \Omega$ with $\gamma_n(a_n) = p_n$ and $\gamma_n(b_n) = q_n$ for all $n \in \mathbb{Z}_+$—such that $\max_{a_n\leq t\leq b_n} \delta_\Omega(\gamma_n(t)) \to 0$ as $n \to \infty$. By assumption, there exist $p' \in \partial\Omega$ and $r > 0$ such that $p \in B(p', r)$, $q \in \partial\Omega\setminus\overline{B(p', r)}$ and $E \cap \partial B(p', r) = \emptyset$. Since $p_n \to p$ and $q_n \to q$ as $n \to \infty$, we may assume that $p_n \in B(p', r)$ and $q_n \in \Omega\setminus\overline{B(p', r)}$ for all n.

Now, as γ_n is a continuous path from p_n to q_n, we have $\gamma_n([a_n, b_n]) \cap \partial B(p', r) \neq \emptyset$. Let $\alpha_n \in (a_n, b_n)$ be such that $\xi_n := \gamma_n(\alpha_n) \in \partial B(p', r)$, and passing to a subsequence if necessary, we may assume that $\xi_n \to \xi \in \partial\Omega \cap \partial B(p', r)$ as $n \to \infty$. Note that $\xi \in \partial\Omega \setminus E$; hence, by assumption, there exists a neighbourhood U of ξ such that conditions (1), (2) and (3) occurring in the statement of Theorem 1.3 are satisfied. Now observe, since $\mathfrak{M}_{\Omega,\,V}(r) \leq \mathfrak{M}_{\Omega,\,U}(r)$ for any neighbourhood $V \subset U$ of ξ, that, shrinking U if necessary, we may assume that $\overline{U} \cap (E \cup \{p, q\}) = \emptyset$ and that U satisfies the three conditions referred to above.

Let $\epsilon > 0$ be such that $\overline{B(\xi, \epsilon)} \subset U$. Since $\xi_n \to \xi$ as $n \to \infty$, we may assume (without loss of generality) that $\xi_n \in B(\xi, \epsilon)$ for all n. Let

$$\beta_n := \inf\left\{t \in [\alpha_n, b_n] : \gamma_n(t) \in \partial B(\xi, \epsilon)\right\}.$$

By definition of β_n and the fact that $\partial B(\xi, \epsilon)$ is closed, we have $\gamma_n(\beta_n) \in \partial B(\xi, \epsilon)$ and $a_n < \alpha_n < \beta_n < b_n$. For every $n \in \mathbb{Z}_+$, define $\sigma_n := \gamma_n|_{[\alpha_n,\,\beta_n]} : [\alpha_n, \beta_n] \longrightarrow \Omega$. It is easy to see that $\sigma_n([\alpha_n, \beta_n]) \subset \overline{B(\xi, \epsilon)}$ for all n. Note, since σ_n is a restriction of the (λ, κ)-almost-geodesic γ_n, that σ_n is also a (λ, κ)-almost-geodesic, for all n. Moreover, we have $\max_{\alpha_n \leq t \leq \beta_n} \delta_\Omega(\sigma_n(t)) \to 0$ as $n \to \infty$. We observe that $\sigma_n([\alpha_n, \beta_n]) \subset \Omega \cap U$ for all $n \in \mathbb{Z}_+$. From this point on, we argue exactly as in the proof of Theorem 1.5 in [2] (and replace M_Ω by $\mathfrak{M}_{\Omega,\,U}$ in the latter proof) to get the result. □

We now present the proof of Corollary 1.4.

The proof of Corollary 1.4 To prove this corollary we shall use the Extended Visibility Lemma. First, we shall show that given E as in the statement of Corollary 1.4 and $p \neq q \in \partial\Omega$, conditions (a) and (b) in the statement of Theorem 1.3 are satisfied. To this end, consider $E_0 := E_a \cup \{p, q\}$. Then, owing to the finiteness of E_0, there exists an ϵ_0 such that $\overline{B(x, \epsilon_0)} \cap \overline{B(x', \epsilon_0)} = \emptyset$ for all $x \neq x' \in E_0$. Now define

$$E_1 := \big(E \cup \{p, q\}\big) \setminus \big(\cup_{x \in E_a} \overline{B(x, \epsilon_0)}\big).$$

Note that E_1 is a finite set disjoint from the compact set $K := \cup_{x \in E_a} \overline{B(x, \epsilon_0)}$. Therefore there exists an $\epsilon_1 > 0$ such that

- $\overline{B(y, \epsilon_1)} \cap K = \emptyset \;\; \forall y \in E_1$;
- $\overline{B(y, \epsilon_1)} \cap \overline{B(y', \epsilon_1)} = \emptyset \;\; \forall y \neq y' \in E_1$.

We now consider two cases:
Case 1. $p \notin K$.
In this case if we take $p' = p$ and $r = \epsilon_1$ then both the conditions (a) and (b) in Theorem 1.3 are satisfied.
Case 2. $p \in K$.
There exists $x_0 \in E_a$ such that $p \in \overline{B(x_0, \epsilon_0)}$. Consider the following finite collection of mutually disjoint sets

$$\mathcal{B} := \left\{\overline{B(x, \epsilon_0)} : x \in E_a\right\} \cup \left\{\overline{B(y, \epsilon_1)} : y \in E_1\right\}.$$

Choose ϵ_2 such that $\epsilon_2 < \mathsf{dist}(B_1, B_2)/4$ for all $B_1 \neq B_2 \in \mathcal{B}$. Then it follows that $\mathcal{C} := \left\{B(x, \epsilon_0 + \epsilon_2) : x \in E_a\right\} \cup \left\{B(y, \epsilon_1 + \epsilon_2) : y \in E_1\right\}$ is a collection of mutually disjoint sets. Now, if we take $p' = x_0$ and $r = \epsilon_0 + \epsilon_2$ then both the conditions (a) and (b) in Theorem 1.3 are satisfied.

Now take an arbitrary point $q' \in \partial\Omega \setminus E$. By Cho [7, Theorem 1], there exist a neighbourhood U of q' in $\mathbb{C}^d$ and positive numbers c, ϵ such that

$$\forall z \in \Omega \cap U, \;\; \forall v \in \mathbb{C}^d, \;\; \kappa_\Omega(z, v) \geq c\frac{\|v\|}{\delta_\Omega(z)^\epsilon}.$$

Therefore, for $r > 0$ sufficiently small, $\mathfrak{M}_{\Omega,\,U}(r) \leq (1/c)r^{\epsilon}$.

It is also a straightforward consequence of Nikolov and Andreev [14, Theorem 7] that, by shrinking U further if necessary, we may assume that there exist a point $z_0 \in \Omega$ and a real number A such that, putting $f(x) := A + (1/2)\log(x)\ \forall\, x \in (0, \infty)$, we have

$$\forall z \in \Omega \cap U,\ \ \mathsf{k}_{\Omega}(z_0, z) \leq f\big(1/\delta_{\Omega}(z)\big).$$

We note that the estimate on $\mathfrak{M}_{\Omega,\,U}(r)$ derived above also holds for this possibly smaller U. It is now easy to check that all the conditions in Theorem 1.3 are satisfied. Consequently, invoking Theorem 1.3, we conclude that Ω is a visibility domain. □

3.2 Weak visibility and geodesic visibility

Before we present our first result, we need a definition. Given a distance space (X, d) and an arbitrary but fixed point $o \in X$, the *Gromov product* is defined by

$$(x|y)_o := (d(x, o) + d(y, o) - d(x, y))/2\ \ \forall\, x, y \in X.$$

We now present

Proposition 3.1 *Suppose that M is a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$.*

(1) *If M has the visibility property with respect to $(1, \kappa)$-almost-geodesics for some $\kappa > 0$, then, for every $p, q \in \partial M$ with $p \neq q$, $\limsup_{(x,y)\to(p,q)}(x|y)_o < \infty$.*
(2) *If, for every $p, q \in \partial M$ with $p \neq q$, $\limsup_{(x,y)\to(p,q)}(x|y)_o < \infty$ and M is, in addition, complete with respect to its Kobayashi distance, then M is a weak visibility submanifold. Further, M is also a geodesic visibility submanifold.*

Proof *Proof of (1):* the proof of this is very similar to that of Bracci et al. [4, Proposition 2.4]. The only difference is that where the authors of Bracci et al. [4] dealt with *geodesics* in domains, we deal with almost-geodesics in complex submanifolds. Since, apart from this difference and the consequent trivial modifications (in particular, the use of Result 2.9 to provide a reverse triangle inequality for triples of points lying on an almost-geodesic), the proofs are almost identical, we omit the proof.
Proof of (2): the proof of this is very similar to that of Bracci et al. [4, Proposition 2.5]. The only difference is the one pointed out in the proof of (1) above. Since, apart from this difference, the consequent trivial modifications, and the taking into account of certain obvious facts that are straightforward analogues of corresponding facts used in the proof of Bracci et al. [4, Proposition 2.5], the proofs are almost identical, we omit the proof. □

Corollary 3.2 *Let M be as in the above proposition, and suppose that it is complete with respect to its Kobayashi distance. Then: M is a geodesic visibility submanifold $\iff$ M is a weak visibility submanifold $\iff$ M is a $(1, \kappa)$-visibility submanifold for some $\kappa > 0$.*

Proof The proof of this corollary follows from that of Proposition 3.1 once we note (see [4, Proposition 2.5]) that M being a geodesic visibility submanifold is equivalent to the finiteness condition on the Gromov product appearing in Proposition 3.1 (work with real Kobayashi geodesics instead of $(1, \kappa)$-almost-geodesics in Part (1) of Proposition 3.1). □

The hypotheses of the next proposition resemble a few of those in Theorem 1.3. The proposition provides a sufficient condition weaker than the one occurring in Proposition 3.1 for a submanifold to be a weak visibility submanifold.

Proposition 3.3 *Let M be as above. Let $E \subset \partial M$ be a closed set such that for any $p \neq q \in \partial M$, there exist $p' \in \partial M$ and $r > 0$ such that $p \in B(p', r)$, $q \in \partial M \setminus \overline{B(p', r)}$ and $E \cap \partial B(p', r) = \emptyset$. Suppose that for some (hence any) $o \in M$, $\mathsf{k}_M(z, o) \to \infty$ as $z \to \xi \in \partial M \setminus E$. Suppose also that*

$$\forall\, p, q \in \partial M \setminus E \text{ with } p \neq q, \quad \limsup_{(x,y)\to(p,q)} (x|y)_o < \infty.$$

Then M is a weak visibility submanifold. When (M, k_M) is complete, M is a geodesic visibility submanifold.

Proof The proof is similar to that of Bracci et al. [4, Proposition 2.5]. The ideas behind the essential modifications required in the proof are the same as those occurring in the proof of Theorem 1.3.

Assume, to get a contradiction, that there exists $\kappa \geq 0$ such that M is not a $(1, \kappa)$-visibility submanifold. Then there exist $p, q \in \partial M$, $p \neq q$, sequences $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ in M such that $p_n \to p$ and $q_n \to q$ as $n \to \infty$, and a sequence $(\gamma_n)_{n\geq 1}$ of $(1, \kappa)$-almost-geodesics, $\gamma_n : [a_n, b_n] \longrightarrow M$, such that $\gamma_n(a_n) = p_n$, $\gamma_n(b_n) = q_n$ for all $n \in \mathbb{Z}_+$ and such that

$$\max_{a_n \leq t \leq b_n} \delta_M(\gamma_n(t)) \to 0 \ \text{ as } \ n \to \infty.$$

By hypothesis, there exist $p' \in \partial M$ and $r > 0$ such that $p \in B(p', r)$, $q \notin \overline{B(p', r)}$ and such that $\partial B(p', r) \cap E = \emptyset$. We now use the arguments in the proof of Theorem 1.3 and those needed to complete that of Proposition 3.1 (part (2)) sketched above to conclude that there exist $\alpha_n, \beta_n, a_n < \alpha_n < \beta_n < b_n$, a point $\xi \in \partial B(p', r) \cap \partial M$, a neighbourhood U of ξ, and $t_n \in [\alpha_n, \beta_n]$ such that $\overline{U} \cap E = \emptyset$ and such that, writing $\sigma_n := \gamma_n|_{[\alpha_n, \beta_n]}$, $\xi_n := \sigma_n(\alpha_n)$, $\eta_n := \sigma_n(\beta_n)$ and $w_n := \sigma_n(t_n)$, we have

- $\sigma_n([\alpha_n, \beta_n]) \subset U$, $\xi_n \to \xi$ as $n \to \infty$ and η_n converges to some point $\eta \neq \xi$ of $U \cap \partial M$;
- for all $n \in \mathbb{Z}_+$, $\|\xi_n - w_n\| = \|\eta_n - w_n\|$ and $(w_n)_{n\geq 1}$ converges to some point w of $U \cap \partial M$ that satisfies $\|w - \xi\| = \|w - \eta\|$.

Since ξ, w and η are all distinct points of $\partial M \setminus E$ and $(\xi_n)_{n\geq 1}$, $(w_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$ converge to ξ, w and η, respectively, therefore, by hypothesis, there exists $C < \infty$ such that

$$\limsup_{n\to\infty} (\xi_n|w_n)_o \leq C \quad \text{and} \quad \limsup_{n\to\infty} (w_n|\eta_n)_o \leq C.$$

Therefore we may, without loss of generality, suppose that there exists $C < \infty$ such that, for all $n \in \mathbb{Z}_+$, $2(\xi_n|w_n)_o \leq C$ and $2(w_n|\eta_n)_o \leq C$. From this point on, we argue exactly as in the concluding part of the proof of Bracci et al. [4, Proposition 2.5] (replacing geodesics by almost-geodesics and using Result 2.9 to obtain reverse triangle inequalities where needed) to obtain the contradiction that $\limsup_{n\to\infty} \mathsf{k}_M(w_n, o) < \infty$ (recall that $(w_n)_{n\geq 1}$ converges to $w \in \partial M \setminus E$). This contradiction shows that M must be a weak visibility submanifold. It is also clear (we simply work with geodesics instead of $(1, \kappa)$-almost-geodesics) that, when (M, k_M) is complete, it is a geodesic visibility submanifold. □

Remark 3.4 The proof above actually shows that under the hypotheses of the above proposition, M satisfies the visibility property with respect to continuous $(1, \kappa)$-quasi-geodesics.

Before we state the next corollary, we need two definitions. The first generalizes Definition 1.11 to the case of convex domains whose boundaries are not necessarily smooth (it is not difficult to check that the following definition is consistent with Definition 1.11).

Definition 3.5 Given a convex domain $\Omega \subset \mathbb{C}^d$, a boundary point p of Ω is said to be *a $\mathbb{C}$-strictly convex boundary point* if for every complex affine line L such that $L \cap \Omega = \emptyset$ and such that $p \in L$, $(L \cap \partial\Omega) \setminus \{p\} = \emptyset$.

We now give the following definition, which we have adopted from Bracci et al. [4] (see [4, Definition 6.12]).

Definition 3.6 Given a domain $\Omega \subset \mathbb{C}^d$, a boundary point p of Ω is said to be *locally $\mathbb{C}$-strictly convex* if there exists a bounded $\mathbb{C}^d$-neighbourhood U of p and a biholomorphism $\Psi : U \to \Psi(U)$ such that $\Psi(U \cap \Omega)$ is a convex domain and such that $\Psi(p)$ is a $\mathbb{C}$-strictly convex boundary point of $\Psi(U \cap \Omega)$.

We are now ready to state and prove the following corollary.

Corollary 3.7 *Let Ω be a bounded domain in $\mathbb{C}^d$. Let $E \subset \partial\Omega$ be a closed set such that for any $p, q \in \partial\Omega$, $p \neq q$, there exist $p' \in \partial\Omega$ and $r > 0$ such that $p \in B(p', r)$, $q \in \partial\Omega \setminus \overline{B(p', r)}$ and $E \cap \partial B(p', r) = \emptyset$. Further, assume that every $q' \in \partial\Omega \setminus E$ is both locally $\mathbb{C}$-strictly convex and a $\mathcal{C}^{1,\,\mathrm{Dini}}$-smooth boundary point. Then Ω is a weak visibility domain. Further, if $(\Omega, \mathsf{k}_\Omega)$ is complete, Ω is a geodesic visibility domain.*

Proof By Proposition 3.3, if we can show that $\mathsf{k}_\Omega(o, z) \to \infty$ as z tends to an arbitrary point of $\partial\Omega \setminus E$ and that, for every $p, q \in \partial\Omega \setminus E$ with $p \neq q$, $\limsup_{(x,y)\to(p,q)} (x|y)_o < \infty$, the result will be proved.

Firstly note that, since every $p \in \partial\Omega \setminus E$ is locally $\mathbb{C}$-strictly convex, every such point is also, by Bracci et al. [4, Theorem 6.13], and to use the terminology of Bracci et al. [4, Definition 6.1], a k-point. This means that

(*) $\forall$ neighbourhood W of p, $\liminf_{z\to p} \big(\mathsf{k}_\Omega(z, \Omega \setminus W) - (1/2)\log\big(1/\delta_\Omega(z)\big) > -\infty$.

In particular, $\lim_{z\to p} \mathsf{k}_\Omega(o, z) = \infty$. But (*) also implies (see [4, Theorem 6.13]) that, if p and q are a pair of distinct points in $\partial\Omega \setminus E$, then they satisfy the log-estimate (see [4, Eq. (2.5)]), i.e., there exist neighbourhoods V and W of p and q, respectively, in $\mathbb{C}^d$, and $C < \infty$ such that, for every $x \in V \cap \Omega$ and every $y \in W \cap \Omega$,

$$\mathsf{k}_\Omega(x, y) \geq (1/2)\log\big(1/\delta_\Omega(x)\big) + (1/2)\log\big(1/\delta_\Omega(y)\big) - C. \tag{3.1}$$

Further, it is a straightforward consequence of Nikolov and Andreev [14, Theorem 7] that we may choose V and W to be so small that there exists $C_1 < \infty$ such that, for every $x \in V \cap \Omega$ and every $y \in W \cap \Omega$,

$$\begin{aligned} \mathsf{k}_\Omega(o, x) &\leq (1/2)\log\big(1/\delta_\Omega(x)\big) + C_1 \quad \text{and} \\ \mathsf{k}_\Omega(o, y) &\leq (1/2)\log\big(1/\delta_\Omega(y)\big) + C_1. \end{aligned}$$

Adding the two inequalities above, we obtain:

$$\mathsf{k}_\Omega(o, x) + \mathsf{k}_\Omega(o, y) \leq (1/2)\log\big(1/\delta_\Omega(x)\big) + (1/2)\log\big(1/\delta_\Omega(y)\big) + 2C_1.$$

Combining the inequality above with (3.1), we get $2(x|y)_o \leq 2C_1 + C$. This shows that $\limsup_{(x,y)\to(p,q)} (x|y)_o < \infty$. By this, the fact that $\lim_{z\to p} \mathsf{k}_\Omega(o, z) = \infty$, and the remark made at the beginning of the proof, the proof of the corollary is complete. □

3.3 Comparison between visibility and geodesic visibility

Let M be a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$ such that (M, k_M) is complete. Suppose that M possesses the visibility property. Then, in particular, it possesses the weak visibility property. Corollary 3.2 then implies that M possesses the geodesic visibility property.

The following proposition shows that in the presence of Gromov hyperbolicity of (M, k_M), visibility and geodesic visibility are equivalent.

Proposition 3.8 *Suppose that M is a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$ such that (M, k_M) is a complete Gromov hyperbolic distance space. Then M is a visibility submanifold if and only if it is a geodesic visibility submanifold.*

Proof That visibility implies geodesic visibility (in the presence of completeness) is clear as argued above. Note that we do not need Gromov hyperbolicity for this implication.

Conversely, in case (M, k_M) is complete and Gromov hyperbolic, the Geodesic Stability Theorem [5, Chapter III.H, Theorem 1.7] (which states, roughly speaking, that in Gromov hyperbolic spaces geodesics and quasi-geodesics are Hausdorff-uniformly close) implies easily that if M satisfies the visibility property with respect to geodesics, then it also satisfies the visibility property with respect to all (λ, κ)-quasi-geodesics, and hence, in particular, that it is a visibility submanifold. □

Remark 3.9 Bharali–Zimmer constructed convex Goldilocks domains that are not Gromov hyperbolic (see, for example [3, Lemma 2.9]). In the next section, we construct two examples of convex domains that are not Goldilocks but that possess versions of the visibility property. Namely, the first example possesses the geodesic visibility property, whereas the second one possesses the (full-fledged) visibility property.

4 Two examples

In this section, we present two examples of bounded convex domains that are not Goldilocks domains; more precisely, condition (1) in Definition 1.2 is not satisfied for either domain. The domain in the first example is a weak visibility domain, while the domain in the second example is a visibility domain. We emphasize that it does **not** seem to be easy to either prove or disprove that the domain in the first example satisfies the visibility property.

4.1 Example of a weak visibility domain that does not satisfy condition (1) in Definition 1.2.

Consider $\Phi_0 : \mathbb{C}^2 \longrightarrow \mathbb{R}$ defined by

$$\Phi_0(z) := \begin{cases} \exp\big(-1/|z_1|^2\big) - \mathsf{Im}(z_2), & \text{if } z_1 \neq 0, \\ -\mathsf{Im}(z_2), & \text{if } z_1 = 0. \end{cases}$$

There exists an $\epsilon > 0$ such that Φ_0 is convex in the ball $B(0, 2\epsilon)$ (in fact, any $\epsilon < 1/\sqrt{6}$ will work). We now choose a $\mathcal{C}^\infty$ function $\psi : \mathbb{C}^2 \longrightarrow [0, 1]$ such that $\psi \equiv 1$ on $B(0, 2\epsilon)$ and such that $\mathsf{supp}\, \psi \subset B(0, 3\epsilon)$. We let $\Phi := \Phi_0 \cdot \psi$ and we also let $c_0 := \sup_{z\in\mathbb{C}^2} \big(-\Phi(z)\big) = \sup_{B(0,3\epsilon)} \big(-\Phi(z)\big) > 0$.

For $n \geq 3$, we consider $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\chi(t) = (t - \epsilon^2)^n$ for all $t > \epsilon^2$ and 0 otherwise. Let $c_1 := \inf_{t \geq (3\epsilon/2)^2} \chi(t)$, and set $C := c_0/c_1$. Define

$$\Psi(z) := C\,\chi(\|z\|^2) \ \ \forall z \in \mathbb{C}^2;$$

and observe:

- Ψ is a $\mathcal{C}^2$-smooth non-negative, convex function on $\mathbb{C}^2$ that is equal to zero on $\overline{B(0, \epsilon)}$, strongly convex locally and strictly positive on $\mathbb{C}^2 \setminus \overline{B(0, \epsilon)}$.
- $\Psi(z) \geq c_0$ for all $z \in \mathbb{C}^2 \setminus \big(B(0, 3\epsilon/2)\big)$. Hence $\Psi(z) + \Phi(z) \geq 0\, \forall z \in \mathbb{C}^2 \setminus \big(B(0, 3\epsilon/2)\big)$.
- For any $z \in B(0, \epsilon)$, $\Psi(z) + \Phi(z) = \Phi(z) = \Phi_0(z)$.

Now consider the domain

$$\Omega := \big\{(z_1, z_2) \in \mathbb{C}^2 : \rho(z) := \Psi(z) + \Phi(z) < 0\big\}.$$

Note that $\Omega \subset B(0, 3\epsilon/2)$, where $\rho = \Psi + \Phi_0$ is convex; consequently, Ω is a bounded convex domain. By computing the gradient of ρ, we see that there exists at most one point $p_0 \in \partial\Omega$ where the gradient vanishes, and this point is of the form $p_0 = (0, ic)$. Moreover, $p_0 \in \overline{B(0, 3\epsilon/2)} \setminus \overline{B(0, \epsilon)}$. It follows then that Ω is a bounded convex domain such that $\partial\Omega \setminus \{p_0\}$ is at least $\mathcal{C}^2$-smooth. It is also clear that any point $x \in \big(\partial\Omega \setminus \{p_0\}\big) \cap \big(\overline{B(0, 3\epsilon)} \setminus \overline{B(0, \epsilon)}\big)$ is a strongly convex boundary point of Ω.

Ω does not satisfy condition (1) in Definition 1.2. We show formally that Ω does not satisfy condition (1) in the definition of a Goldilocks domain. So what we need to do is show that for every $\epsilon_0 > 0$ sufficiently small, $\int_0^{\epsilon_0} (\mathfrak{M}_\Omega(r)/r) dr = \infty$. The way our domain Ω has been defined, there exists $r_0 > 0$ such that

$$\Omega \cap B(0, r_0) = \big\{(z_1, z_2) \in B(0, r_0) : \mathsf{Im}(z_2) > \exp(-1/|z_1|^2)\big\}.$$

Fix $\epsilon_0 \in (0, r_0)$. It is immediate that for every $r \in (0, \epsilon_0)$, $\delta_\Omega\big((0, ir)\big) \leq r$ (consider the boundary point $0_{\mathbb{C}^2}$ of Ω). Now we use the elementary upper bound on the Kobayashi metric to write

$$\kappa_\Omega\big((0, ir); (1, 0)\big) \leq 1/\Delta_\Omega\big((0, ir); (1, 0)\big),$$

where $\Delta_\Omega(z; v) := \sup\big\{t > 0 \mid \big(z + (t\mathbb{D})(v/\|v\|)\big) \subset \Omega\big\}$. From the explicit description of $\Omega \cap B(0, r_0)$, it follows that $\Delta_\Omega\big((0, ir); (1, 0)\big) = 1/\sqrt{\log(1/r)}$. Therefore

$$\mathfrak{M}_\Omega(r) \geq \frac{1}{\kappa_\Omega\big((0, ir); (1, 0)\big)} \geq \Delta_\Omega\big((0, ir); (1, 0)\big) = 1/\sqrt{\log(1/r)}.$$

Since

$$\int_0^{\epsilon_0} \frac{dr}{r\sqrt{\log(1/r)}} = \infty,$$

Ω is *not* a Goldilocks domain.

Every point of $\partial\Omega$ except possibly p_0 is $\mathbb{C}$-strictly convex. Consider

$$S := \partial\Omega \cap \overline{B(0, \epsilon)} = \overline{B(0, \epsilon)} \cap \big\{z \in \mathbb{C}^2 : \Phi_0(z) = 0\big\}. \tag{4.1}$$

Then, as noticed earlier, any $p \in \partial\Omega \setminus S$, $p \neq p_0$, is a strongly convex, and therefore also a $\mathbb{C}$-strictly convex boundary point. Next, we shall show that any $p \in S$ is also a $\mathbb{C}$-strictly convex boundary point. This will establish that every point of $\partial\Omega$ except possibly p_0 is a

$\mathbb{C}$-strictly convex boundary point. An easy computation, taking into account the fact that $\Psi \equiv 0$ on $B(0, \epsilon)$, shows that, for all $p \in S$,

$$\begin{aligned} H_p(\partial\Omega) &= \Big\{\xi = (\xi_1, \xi_2) \in \mathbb{C}^2 : s(p_1)\bar{p}_1\xi_1 - (1/2i)\xi_2 = 0\Big\} \\ &= \mathsf{span}_{\mathbb{C}}\big\{(1,\ 2is(p_1)\bar{p}_1)\big\}, \end{aligned} \tag{4.2}$$

where $s(p_1) := \exp(-1/|p_1|^2)/|p_1|^4$ for all $p_1 \neq 0$, and where $s(0) := 0$. (In particular, when $p_1 = 0$, $H_p(\partial\Omega) = \mathsf{span}_{\mathbb{C}}\big\{(1, 0)\big\} = \mathbb{C} \times \{0\}$.) Write $u(p) := 2is(p_1)\bar{p}_1$ for all $p \in S$. Then, for every $p \in S$, $H_p(\partial\Omega) = \mathsf{span}_{\mathbb{C}}\big\{(1, u(p))\big\}$. From this it follows that Ω fails to be $\mathbb{C}$-strictly convex at some point of S if and only if there exist $p \in S$ and $\zeta \in \mathbb{C}\setminus\{0\}$ such that $p + \zeta(1, u(p)) \in \partial\Omega$. From this it follows easily that

$$\forall\, t \in [0, 1],\ \ p + t\zeta(1, u(p)) \in S.$$

Since S is as given in (4.1), this implies that

$$\forall\, t \in [0, 1],\ \ \Phi_0\big(p + t\zeta(1, u(p))\big) = 0.$$

But, writing down the definition of Φ_0 and recalling that $\zeta \neq 0$, we see that this yields an immediate contradiction. From this contradiction it follows that Ω is $\mathbb{C}$-strictly convex at every point of S, hence (recalling what we observed previously) that Ω is $\mathbb{C}$-strictly convex at every boundary point except, possibly, p_0.

Now, since Ω also has at least $\mathcal{C}^2$-smooth boundary, we see that we may appeal to Corollary 3.7 to conclude that Ω is a weak visibility domain. We emphasize that it is unclear whether Ω is a visibility domain. The reason is that all boundary points of the type $(0, t)$ with t real, $|t| < \epsilon$, are points of infinite type, as can easily be checked. Therefore we cannot invoke any known theorem to conclude visibility.

4.2 Example of a domain that satisfies the condition in Corollary 1.4 but does not satisfy condition (1) in Definition 1.2

Consider $\Phi_0 : \mathbb{C}^2 \longrightarrow \mathbb{R}$ defined by

$$\Phi_0(z) := \begin{cases} \exp\big(-1/\|z\|^2\big) - \mathsf{Im}(z_2), & z \neq 0, \\ 0, & z = 0, \end{cases}$$

where $\|z\|$ denotes the Euclidean norm of $z \in \mathbb{C}^2$. There exists an $\epsilon > 0$ such that Φ_0 is convex in the ball $B(0, 2\epsilon)$ (in fact, any $\epsilon < 1/2\sqrt{2}$ will work). We now choose a $\mathcal{C}^\infty$ function $\psi : \mathbb{C}^2 \longrightarrow [0, 1]$ such that $\psi \equiv 1$ on $B(0, 2\epsilon)$ and such that $\mathsf{supp}\,\psi \subset B(0, 3\epsilon)$. We let $\Phi := \Phi_0 \cdot \psi$ and we also let $c_0 := \sup_{z\in\mathbb{C}^2}\big(-\Phi(z)\big) = \sup_{B(0,3\epsilon)}\big(-\Phi(z)\big) > 0$.

We choose another function $\chi : [0, \infty) \longrightarrow [0, \infty)$ that is (1) $\mathcal{C}^\infty$, (2) identically 0 on $[0, \epsilon^2]$ and (3) strictly increasing on $[\epsilon^2, \infty)$ and strongly convex on $(\epsilon^2, (\epsilon + \delta)^2)$ (that is, has double derivative positive) for some small $\delta > 0$ (for example, one could take $\chi = \exp\big(-1/(t - \epsilon^2)\big)$ when $t > \epsilon^2$ and 0 otherwise). Let $c_1 := \inf_{t\geq(\epsilon+\delta/2)^2} \chi(t)$, and set $C := c_0/c_1$. Define

$$\Psi(z) := C\,\chi(\|z\|^2)\ \ \forall\, z \in \mathbb{C}^2;$$

and observe:

- Ψ is a $\mathcal{C}^\infty$-smooth, non-negative function on $\mathbb{C}^2$ that is equal to zero on $\overline{B(0, \epsilon)}$ and that is strongly convex and strictly positive on $B(0, \epsilon + \delta)\setminus\overline{B(0, \epsilon)}$.

- $\Psi(z) \geq c_0$ for all $z \in \mathbb{C}^2 \setminus B(0, \epsilon + \delta/2)$. Hence $\Psi(z) + \Phi(z) \geq 0\ \forall z \in \mathbb{C}^2 \setminus B(0, \epsilon + \delta/2)$.
- For any $z \in B(0, \epsilon)$, $\Psi(z) + \Phi(z) = \Phi(z) = \Phi_0(z)$.

Now consider the domain

$$\Omega := \{z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) := \Psi(z) + \Phi(z) < 0\}.$$

Note that $\Omega \subset B(0, \epsilon + \delta/2)$, on which $\rho = \Psi + \Phi_0$ is convex; consequently, Ω is a bounded convex domain. By computing the gradient of ρ, we see that there exists at most one point $p_0 \in \partial\Omega$ where the gradient vanishes, and this point is of the form $p_0 = (0, ic)$. Moreover, $p_0 \in \overline{B(0, \epsilon + \delta/2)} \setminus \overline{B(0, \epsilon)}$. It follows then that Ω is a bounded convex domain such that $\partial\Omega \setminus \{p_0\}$ is $\mathcal{C}^\infty$-smooth. It is also clear that any point $x \in (\partial\Omega \setminus \{p_0\}) \cap (\overline{B(0, \epsilon + \delta/2)} \setminus \overline{B(0, \epsilon)})$ is a strongly convex boundary point of Ω, whence it is of finite type. Set $S := \partial\Omega \cap \overline{B(0, \epsilon)}$ and observe that

$$S = \overline{B(0, \epsilon)} \cap \{z \in \mathbb{C}^2 : \Phi_0(z) = 0\}. \tag{4.3}$$

It is also easy to show that any point in S different from 0 is a boundary point of Ω of finite type. Hence, using Corollary 1.4, it follows that Ω is a visibility domain. That Ω is a geodesic visibility domain follows easily from Corollary 3.2

Ω does not satisfy condition (1) in Definition 1.2. Note that

$$\Omega \cap B(0, \epsilon/2) = \{(z_1, z_2) \in B(0, \epsilon/2) : \mathsf{Im}(z_2) > \exp\left(-1/\|z\|^2\right)\}.$$

From the above expression, we see that, for r sufficiently small, $p_r := (0, ir) \in \Omega$. Write $v := (1, 0)$; we regard v as a unit vector in $\mathbb{C}^2$. It is easy to see that $\Delta_\Omega(p_r, v) \geq \rho$, where ρ is given by $\rho = \sqrt{(1/\log(1/r)) - r^2}$. Therefore, using arguments similar to those used in dealing with Example 4.1, we readily obtain $\mathfrak{M}_\Omega(r) \geq 1/\kappa_\Omega(p_r, v) \geq \rho = \sqrt{(1/\log(1/r)) - r^2}$. Therefore, to prove that Ω does not satisfy Condition 1 in the definition of a Goldilocks domain, it suffices to show that for $\delta > 0$ sufficiently small so that the integrand makes sense,

$$\int_0^\delta \frac{1}{r}\sqrt{\frac{1}{\log(1/r)} - r^2}\, dr = \infty.$$

This follows easily. Therefore Ω is not a Goldilocks domain.

5 Properties of Visibility Subspaces and the Continuous Extension of Kobayashi Isometries

In this section we shall make the requisite comments about the proof of Theorem 1.9, and also prove Theorem 1.10 and related corollaries. In the first subsection below, we make certain observations regarding geodesic subspaces and also present two lemmas about visibility subspaces, which will be needed in the proofs of the aforementioned theorems. In the next subsection, we deal with the proofs proper.

As promised in the Introduction, we first provide a sketch of an argument showing why every subspace V_f of $\mathbb{D}^n$, $n \geq 2$, of the form $V_f = \{(z, f(z)) : z \in \mathbb{D}\}$, where $f = (f_1, \ldots, f_{n-1}) : \mathbb{D} \longrightarrow \mathbb{D}^{n-1}$ is a holomorphic map, is a visibility subspace if f extends continuously to $\overline{\mathbb{D}}$. It is easy to see that every point of $\partial_a V_f = \overline{V_f} \setminus V_f$ is of the form $(\zeta, f(\zeta))$, where $|\zeta| = 1$. Using this fact, the visibility property of $\mathbb{D}$, the explicit form of

$\mathsf{k}_{\mathbb{D}^n}$, and the distance-decreasing property of holomorphic maps with respect to the Kobayashi distance, it is now easy to show that any pair of distinct points of $\partial_a V_f$ satisfies the visibility property with respect to geodesics of $\mathsf{k}_{\mathbb{D}^n}$.

5.1 Preliminary observations regarding geodesic subspaces and two preparatory lemmas

Given a geodesic subspace S of (M, k_M), it is easy to see from the definition that S is closed in M, that $(S, \mathsf{k}_M|_{S\times S})$ is locally compact, and that the other crucial hypothesis in [11, Theorem 8.1] is satisfied. Therefore, by this latter result, the completeness of $(S, \mathsf{k}_M|_{S\times S})$ is equivalent to the condition that every closed ball is compact, i.e., the distance space is proper. In particular, if z_0 is any fixed point of S, if p is a fixed but arbitrary point of $\partial_a S := \overline{S} \setminus S$, and if $(z_n)_{n\geq 1}$ is a sequence in S converging, in the Euclidean sense, to p, then $\lim_{n\to\infty} \mathsf{k}_M(z_0, z_n) = \infty$. This latter fact also implies that $\partial_a S \subset \partial M = \overline{M}\setminus M$.

Let $M = D$, a bounded domain in $\mathbb{C}^d$. If (D, k_D) is a complete distance space, then any closed subset S of D satisfies the first defining condition of a geodesic subspace. Thus, in this case, every closed subset S of D that satisfies only the second condition in Definition 1.7 will be a geodesic subspace. For example: every holomorphic retract of a complete distance space (D, k_D) is a geodesic subspace. Let Ω and D be bounded domains in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, such that $(\Omega, \mathsf{k}_\Omega)$ is complete. Let $f : \Omega \longrightarrow D$ be an isometry with respect to the Kobayashi distances (we are not making any claims about the existence of such isometries). If we write $S := f(\Omega) \subset D$, then it is easy to see that S is a geodesic subspace of D. This example suggests that there could be a bounded domain D such that (D, k_D) is not complete, but such that D nevertheless has geodesic subspaces. Indeed, this is the case.

Example 5.1 Let $\Omega \subset \mathbb{C}^d$ be a bounded convex domain and let A be an analytic subset of Ω of co-dimension at least 2. Let $D := \Omega \setminus A$. Then $\mathsf{k}_D = \mathsf{k}_\Omega|_{D\times D}$ (see, for example, [6, Theorem 2]). Choose a complex geodesic f in Ω that avoids A. Clearly, f is a complex geodesic in D too. Note that (D, k_D) is not complete. But $f(\mathbb{D})$ is a geodesic subspace of D.

We need a definition. Before providing it, we clarify that, if X is a given topological space, then by a compactification of X we shall mean a pair $(\iota, \widetilde{X})$, where $\widetilde{X}$ is required to be a compact Hausdorff topological space and $\iota : X \longrightarrow \widetilde{X}$ is required to be a homeomorphism onto its image $\iota(X)$, which is, in addition, required to be an open, dense subset of $\widetilde{X}$. We shall regard X as being a subset of $\widetilde{X}$ (by identifying X with $\iota(X)$).

Definition 5.2 Let (X, d) be a proper, geodesic distance space and let $(\iota, \widetilde{X})$ be a compactification of X (regarded as a topological space with the topology induced by d). By a *geodesic loop of X in $\widetilde{X}$* we mean a geodesic line γ in (X, d) (that is, an isometric embedding γ from $(\mathbb{R}, |\cdot|)$ to (X, d)) such that the set of limit points of γ at ∞ is equal to the set of limit points of γ at $-\infty$. (Note that the set of limit points of γ at ∞ (and $-\infty$) is contained in $\widetilde{X}\setminus X$.)

We point out that we will only use this notion in the case where $X = S$ is a geodesic subspace of a bounded, connected, embedded submanifold M of $\mathbb{C}^d$.

We note that it is easy to define the notion of visibility for a pair consisting of a proper geodesic distance space (X, d) and a compactification $(\iota, \widetilde{X})$ of X, by analogy with Definition 1.8. The important thing for us to note is that if X has the visibility property with respect to the compactification $\widetilde{X}$, the proof of the first part of [4, Lemma 3.1] goes through without change to show that every geodesic ray γ in (X, d) (i.e., an isometric embedding $\gamma : \big([0, \infty), |\cdot|\big) \longrightarrow (X, d)$) *lands* at a point of $\widetilde{X}\setminus X$, i.e., $\lim_{t\to\infty} \gamma(t)$ exists as an element of $\widetilde{X} \setminus X$ (which is the boundary of X in $\widetilde{X}$). We note that, in such a situation, a geodesic loop of X in $\widetilde{X}$ is a geodesic line γ such that $\lim_{t\to-\infty} \gamma(t) = \lim_{t\to\infty} \gamma(t)$.

We now state two lemmas, the second of which is a mild generalization of Bracci et al. [4, Lemma 3.1] and which was referred to above. The utility of these lemmas will become apparent when we prove Theorem 1.9. Since the proof of the second lemma is substantially the same as that of [4, Lemma 3.1], we omit the proof. The essential observation here is that the proof in [4] goes through virtually without modification in the more general setting of visibility subspaces.

Lemma 5.3 *Suppose that $M \subset \mathbb{C}^d$ is a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$ and that S is a visibility subspace of M. If $(z_\nu)_{\nu\geq1}$ and $(w_\nu)_{\nu\geq1}$ are sequences in S converging to* distinct *boundary points $p, q \in \partial_a S$, then $\mathsf{k}_M(z_\nu, w_\nu) \to \infty$ as $\nu \to \infty$.*

Proof Note that the proof of (1) of Proposition 3.1 goes through with almost no modifications to show that $\limsup_{\nu\to\infty}(z_\nu|w_\nu)_o < \infty$, where the Gromov product is now calculated with respect to $\mathsf{k}_M|_{S\times S}$ and for any fixed $o \in S$. Combining this with our previous observation that $\mathsf{k}_M(x, o) \to \infty$ when $S \ni x \to x_0 \in \partial_a S$, the required conclusion follows immediately. □

Lemma 5.4 *Let M and S be as above. Then any geodesic ray γ in S lands at a point p of $\partial_a S$, i.e., there exists $p \in \partial_a S$ such that $\lim_{t\to\infty} \gamma(t) = p$.*

Conversely, suppose that $z_0 \in S$ and that $(z_\nu)_{\nu\geq1}$ is a sequence in S converging to a point $p \in \partial_a S$. For every ν, let γ_ν be a k_M-geodesic in S joining z_0 to z_ν. Then, up to a subsequence, $(\gamma_\nu)_{\nu\geq1}$ converges uniformly on the compact subsets of $[0, \infty)$ to a geodesic ray that lands at p.

5.2 The proofs of Theorems 1.9 and 1.10

With these two lemmas in place, it is easy to see that one can, without any difficulty, replicate the arguments in the proof of [4, Theorem 3.3] to prove Theorem 1.9. We therefore omit the proof of the latter.

We now illustrate the usefulness of Theorem 1.9 by proving Theorem 1.10.

The proof of Theorem 1.10 Note that f is an isometry between (M, k_M) and $(S, \mathsf{k}_N|_{S\times S})$. Therefore, $(S, \mathsf{k}_N|_{S\times S})$ is Gromov hyperbolic (because (M, k_M) is by assumption so). By the general theory of Gromov hyperbolic spaces (see [5, Part III, Chapter H, Theorem 3.9]), f extends to a homeomorphism $\widetilde{f}$ from $\overline{M}^G$ to $\overline{S}^G$. By Theorem 1.9, $\mathsf{id}_S : S \longrightarrow S$ extends to a continuous surjection $\widehat{\mathsf{id}}_S : \overline{S}^G \longrightarrow \overline{S}$. There is also a natural inclusion $i_{\overline{S}}$ of $\overline{S}$ in $\overline{N}$. If we define $\widehat{f} := i_{\overline{S}} \circ \widehat{\mathsf{id}}_S \circ \widetilde{f}$, then it is clear that $\widehat{f} : \overline{M}^G \longrightarrow \overline{N}$ is a continuous extension of f. If S has no geodesic loops in $\overline{S}$ then, again by Theorem 1.9, $\widehat{\mathsf{id}}_S$ is a homeomorphism from $\overline{S}^G$ to $\overline{S}$ and it follows from the definition of $\widehat{f}$ that, regarding it as a mapping from $\overline{M}^G$ to $\overline{S}$, it is a homeomorphism. □

Now, we shall present two important corollaries of Theorem 1.10.

Corollary 5.5 *Suppose that $M \subset \mathbb{C}^m$ and $N \subset \mathbb{C}^n$ are bounded, connected, embedded complex submanifolds and that M is complete with respect to its Kobayashi distance. Suppose that $f : M \longrightarrow N$ is an isometry with respect to the Kobayashi distances. Suppose that (M, k_M) is Gromov hyperbolic and that N is a weak visibility submanifold. Then f extends* *to a continuous map* $\widehat{f} : \overline{M}^G \longrightarrow \overline{N}$*, where* $\overline{M}^G$ *denotes the Gromov compactification of* (M, k_M).

Proof Since N is a weak visibility submanifold, (1) of Proposition 3.1 gives us: for every $p, q \in \partial N$, $p \neq q$, $\limsup_{(x,y)\to(p,q)}(x|y)_o < \infty$. In particular,

$$\forall\, p, q \in \partial_a S \text{ with } p \neq q, \quad \limsup_{S\times S\ni(x,y)\to(p,q)} (x|y)_o < \infty,$$

where we take $S := f(M)$, and o to be an arbitrary but fixed point of S. Now the reader can easily verify that precisely the same method that is used to prove (2) of Proposition 3.1 can also be used, in this case, to show that S is a visibility subspace of N (keep in mind that $(S, \mathsf{k}_N|_{S\times S})$ is a *proper* distance space). So we may once again apply Theorem 1.10 to draw the desired conclusion. □

In particular, when $M = \mathbb{D}$, which is a complete, Gromov hyperbolic distance space with respect to the Kobayashi distance $\mathsf{k}_{\mathbb{D}}$ and for which the Gromov compactification is known to coincide with the Euclidean compactification, we have:

Corollary 5.6 *Suppose that* $M \subset \mathbb{C}^m$ *is a weak visibility submanifold. Suppose that* $f : \mathbb{D} \longrightarrow M$ *is a complex geodesic. Then* f *extends to a continuous map* $\widehat{f} : \overline{\mathbb{D}} \longrightarrow \overline{M}$.

We are now ready to present the proof of Corollary 1.12.

The proof of Corollary 1.12 We first note that, by the arguments that occur in the first part of the proof of Corollary 1.4, we can show that the hypotheses of Corollary 3.7 in Sect. 3 are satisfied by Ω. Consequently, by Corollary 3.7, Ω is a weak visibility domain. Now we can invoke Corollary 5.6 to obtain the desired conclusion. □

6 A Wolff–Denjoy-type theorem

In this section we present a proof of Theorem 1.14. Our proof relies on two crucial ingredients that are consequences of visibility with respect to $(1, \kappa)$-almost-geodesics for some $\kappa > 0$. In the first subsection below, we present these ingredients first.

6.1 Preparations

Our first ingredient is an analogue of Proposition 4.1 in [2]. Its proof is based on an argument developed by Karlsson in [10].

Proposition 6.1 *Let* M *be a bounded, connected, embedded complex submanifold of* $\mathbb{C}^d$*. Suppose there exists* $\kappa_0 > 0$ *such that* M *possesses the visibility property with respect to* $(1, \kappa_0)$*-almost-geodesics. Let* $\nu, \mu : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ *be strictly increasing functions such that there exists* $m_0 \in M$ *so that*

$$\lim_{j\to\infty} \mathsf{k}_M\big(F^{\nu(j)}(m_0), m_0\big) = \lim_{j\to\infty} \mathsf{k}_M\big(F^{\mu(j)}(m_0), m_0\big) = \infty. \tag{6.1}$$

Then there exists $\xi \in \partial M$ *such that, for all* $z \in M$*,* $\lim_{j\to\infty} F^{\nu(j)}(z) = \lim_{j\to\infty} F^{\mu(j)}(z) = \xi$.

Proof The proof of Bharali and Maitra [2, Proposition 4.1] goes through without modification. The only observation to be made is that the argument given there works for weak visibility submanifolds, not just visibility domains as considered in [2], when one takes into account the essential lemmas regarding the Kobayashi distance and metric on submanifolds presented in Sect. 2 of this paper. □

Our second ingredient is Theorem 6.6 below, which is a consequence of Theorem 6.4. The latter theorem says that, when M is taut, visibility with respect to $(1, \kappa)$-almost-geodesics for some $\kappa > 0$ implies that $\mathfrak{M}_M(r) \to 0$ as $r \to 0$. (We recall that the notation $\mathfrak{M}_M(r)$ is explained right after (1.1).) To prove Theorem 6.4, we need two lemmas. Both of them are elementary; so we state them here without proof.

Lemma 6.2 *Suppose that $(\phi_\nu)_{\nu\geq 1}$ is a sequence of holomorphic maps from $\mathbb{D}$ to $\mathbb{C}^d$. Suppose that $(\phi_\nu)_{\nu\geq 1}$ is uniformly bounded. Then, for every $r_0 \in (0, 1)$, there exists $L = L(r_0) < \infty$ such that*

$$\forall\, \nu \in \mathbb{Z}_+,\ \forall \zeta_1, \zeta_2 \in D(0, r_0),\ \|\phi_\nu(\zeta_1) - \phi_\nu(\zeta_2)\| \leq L|\zeta_1 - \zeta_2|,$$

where $D(0, r_0) := \{\zeta \in \mathbb{C} : |\zeta| < r_0\}$.

Lemma 6.3 *Suppose that $(\phi_\nu)_{\nu\geq 1}$ is a sequence of holomorphic maps from $\mathbb{D}$ to $\mathbb{C}^d$. Suppose that $(\phi_\nu)_{\nu\geq 1}$ is uniformly bounded and that there exists $\epsilon_0 > 0$ such that, for all $\nu \in \mathbb{Z}_+$, $\|\phi'_\nu(0)\| \geq \epsilon_0$. Then there exists $\delta > 0$ such that*

$$\forall\, \nu \in \mathbb{Z}_+,\ \forall\, \zeta_1, \zeta_2 \in \overline{D(0, \delta)},\ \|\phi_\nu(\zeta_1) - \phi_\nu(\zeta_2)\| \geq (\epsilon_0/2)|\zeta_1 - \zeta_2|,$$

where $D(0, \delta) := \{\zeta \in \mathbb{C} : |\zeta| < \delta\}$.

Theorem 6.4 *Let M be a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$. Suppose that M has the visibility property with respect to $(1, \kappa)$-almost-geodesics for some $\kappa > 0$ and, moreover, that M is taut. Then $\mathfrak{M}_M(r) \to 0$ as $r \to 0$.*

Proof We will closely follow the proof of Theorem 4.2 in [2]. However, we provide a complete proof here because there is an essential difference between the proof of [2, Theorem 4.2] and the current proof, which is that the former *does not* work if it is only known that M possesses the visibility property with respect to $(1, \kappa)$-almost-geodesics for *some* $\kappa > 0$.

Assume, to get a contradiction, that $\mathfrak{M}_M(r) \not\to 0$ as $r \to 0$. Since $\mathfrak{M}_M(r)$ decreases as r decreases to 0, the above assumption implies that there exists $\epsilon_1 > 0$ such that $\mathfrak{M}_M(r) \downarrow \epsilon_1$ as $r \downarrow 0$. Let $\epsilon_0 := \epsilon_1/2$. Then, for every $\nu \in \mathbb{Z}_+$, $\mathfrak{M}_M(1/\nu) > \epsilon_0$. Therefore, for every $\nu \in \mathbb{Z}_+$, there exist $z_\nu \in M$ such that $\delta_M(z_\nu) \leq 1/\nu$ and $v_\nu \in T^{(1,0)}_{z_\nu} M$ with $\|v_\nu\| = 1$ such that $1/\kappa_M(z_\nu, v_\nu) > \epsilon_0$, i.e., $\kappa_M(z_\nu, v_\nu) < 1/\epsilon_0$. We also assume, without loss of generality, that $(z_\nu)_{\nu\geq 1}$ converges to some point $\xi \in \partial M$. By the definition of κ_M, the inequalities above imply that there exist a holomorphic map $\phi_\nu : \mathbb{D} \longrightarrow M$ such that $\phi_\nu(0) = z_\nu$ and a $t_\nu \in (0, 1/\epsilon_0)$ such that $t_\nu \phi'_\nu(0) = v_\nu$. This last equation implies: $t_\nu \|\phi'_\nu(0)\| = 1$, which in turn implies that, for all $\nu \in \mathbb{Z}_+$, $\|\phi'_\nu(0)\| > \epsilon_0$. By the tautness of M, there exists a subsequence of $(\phi_\nu)_{\nu\geq 1}$, which we continue to denote by $(\phi_\nu)_{\nu\geq 1}$ without changing subscripts, that converges uniformly on the compact subsets of $\mathbb{D}$ to a holomorphic map ϕ that is either M-valued or ∂M-valued. Note that

$$\phi(0) = \lim_{\nu\to\infty} \phi_\nu(0) = \lim_{\nu\to\infty} z_\nu = \xi.$$

Therefore, ϕ is ∂M-valued. Note that $\|\phi'(0)\| \geq \epsilon_0$; so ϕ is non-constant.

Now we invoke Lemma 6.3 to conclude that there exists $\delta \in (0, 1)$, $\delta \leq \tanh(\kappa)$, such that

$$\forall\, \nu \in \mathbb{Z}_+,\ \forall\, \zeta_1, \zeta_2 \in \overline{D(0,\delta)},\ \|\phi_\nu(\zeta_1) - \phi_\nu(\zeta_2)\| \geq (\epsilon_0/2)|\zeta_1 - \zeta_2|. \tag{6.2}$$

Define $\eta := \phi(\delta/2) \in \partial M$ and $w_\nu := \phi_\nu(\delta/2)$. Then $(w_\nu)_{\nu\geq 1}$ is a sequence in M converging to η. By (6.2), it follows immediately that $\phi(0) \neq \phi(\delta/2)$, i.e., $\xi \neq \eta$. The sequences $(z_\nu)_{\nu\geq 1}$ and $(w_\nu)_{\nu\geq 1}$ in M converge to $\xi \in \partial M$ and $\eta \in \partial M$, respectively.

Next, we shall show that $\gamma_\nu := \phi_\nu \circ \sigma : [0, R] \longrightarrow M$ is a $(1, \kappa)$-almost-geodesic in M, where $\sigma : [0, R] \longrightarrow \mathbb{D}$ is the geodesic in $\mathbb{D}$ for the Poincaré distance joining 0 to $\delta/2$ (R is, in fact, equal to $\tanh^{-1}(\delta/2)$ and σ itself is just $\tanh|_{[0,\tanh^{-1}(\delta/2)]}$). By the explicit form of σ, there exists $M_\delta > 1$ such that, for every $s, t \in [0, R]$, $(1/M_\delta)|s - t| \leq |\sigma(s) - \sigma(t)| \leq M_\delta|s - t|$. Now note that

$$\begin{aligned}\forall\, \nu \in \mathbb{Z}_+,\ \forall\, s, t \in [0, R], \quad \|\gamma_\nu(s) - \gamma_\nu(t)\| &= \|\phi_\nu(\sigma(s)) - \phi_\nu(\sigma(t))\| \\ &\leq L|\sigma(s) - \sigma(t)| \leq LM_\delta|s - t|.\end{aligned}$$

To write the second inequality above, we use Lemma 6.2. Therefore, all the γ_ν's are Lipschitz (in fact, as we see, LM_δ works as a Lipschitz constant for all of them) and therefore they are absolutely continuous (in fact, each γ_ν is clearly $\mathcal{C}^\infty$-smooth). We compute

$$\begin{aligned}\forall\, \nu \in \mathbb{Z}_+,\ \forall\, t \in [0, R], \quad \kappa_M(\gamma_\nu(t), \gamma_\nu'(t)) &= \kappa_M\big(\phi_\nu(\sigma(t)), \sigma'(t)\phi_\nu'(\sigma(t))\big) \\ &\leq \kappa_{\mathbb{D}}(\sigma(t), \sigma'(t)) = 1.\end{aligned}$$

(To write the second inequality above, we use the fact that ϕ_ν is contractive relative to the Kobayashi metrics; the final equality follows because σ is a geodesic for the Poincaré metric.) We also observe that

$$\begin{aligned}\forall\, \nu \in \mathbb{Z}_+,\ \forall\, s, t \in [0, R], \quad \mathsf{k}_M(\gamma_\nu(s), \gamma_\nu(t)) &= \mathsf{k}_M\big(\phi_\nu(\sigma(s)), \phi_\nu(\sigma(t))\big) \\ &\leq \mathsf{k}_{\mathbb{D}}(\sigma(s), \sigma(t)) = |s - t|.\end{aligned}$$

(To write the second inequality above, we use the fact that ϕ_ν is contractive relative to the Kobayashi distances; the final equality follows because σ is a geodesic for the Poincaré distance.)

Finally, since $|s - t| \leq R \leq \tanh^{-1}(\delta) \leq \tanh^{-1}(\tanh(\kappa)) = \kappa$ for all $s, t \in [0, R]$, we have

$$|s - t| - \kappa \leq 0 \leq \mathsf{k}_M\big(\gamma_\nu(s), \gamma_\nu(t)\big) \leq |s - t| < |s - t| + \kappa$$

for all $s, t \in [0, R]$. The above considerations show that each γ_ν is a $(1, \kappa)$-almost-geodesic. By our assumption that M is a $(1, \kappa)$-visibility submanifold, it follows that there exists a compact subset K of M such that, for every $\nu \in \mathbb{Z}_+$, $\mathsf{ran}(\gamma_\nu) \cap K \neq \emptyset$. But $\mathsf{ran}(\gamma_\nu) \subset \mathsf{ran}(\phi_\nu)$ and, since $(\phi_\nu)_{\nu\geq 1}$ converges uniformly on the compact subsets of $\mathbb{D}$ to a ∂M-valued map, it follows that for every compact subset K of M, $\mathsf{ran}(\gamma_\nu) \cap K = \emptyset$ for all ν sufficiently large. This is a contradiction. So our starting assumption must be wrong, and $\mathfrak{M}_M(r) \to 0$ as $r \to 0$. □

Remark 6.5 Theorem 6.4 shows that, in particular, if a bounded, convex domain Ω is a weak visibility domain, one has $\mathfrak{M}_\Omega(r) \to 0$ as $r \to 0$.

We now prove the following analogue of Theorem 4.3 in [2].

Theorem 6.6 *Suppose that M is a bounded, connected, embedded complex submanifold of $\mathbb{C}^d$. Suppose that M is a $(1, \kappa)$-visibility submanifold for some $\kappa > 0$ and that it is also taut. Suppose that X is a connected complex manifold and that $(\phi_\nu)_{\nu \geq 1}$ is a sequence of holomorphic maps from X to M that converges, uniformly on the compact subsets of X, to a holomorphic map ψ from X to ∂M. Then ψ is constant.*

Proof We argue exactly as in the proof of Theorem 4.3 in [2]. The latter result was a direct consequence of the fact that $\mathfrak{M}_\Omega(r) \to 0$ as $r \to 0$ for a taut domain Ω satisfying the visibility property. The corresponding result in the present case is Theorem 6.4. □

6.2 Proof of Theorem 1.14

We are now ready to give a sketch of the proof of Theorem 1.14.

Proof of Theorem 1.14 We argue exactly as in the proof of Theorem 1.8 in [2], replacing Ω there by M, and consider two subcases as in the latter proof. In the first subcase, the results employed in the latter proof are Bharali and Maitra [2, Theorem 4.3] and Bharali and Maitra [2, Proposition 4.1]. The results analogous to these in this paper are Theorem 6.6 and Proposition 6.1, respectively, which we employ in our argument to settle this subcase.

In the second subcase, the results employed in the proof of Bharali and Maitra [2, Theorem 1.8] are Bharali and Maitra [2, Result 2.1] and [2, Lemma 2.9] and the existence of (λ, κ)-almost-geodesics on bounded domains. The results analogous to these in this paper are Remark 2.3, Theorem 2.8 and Result 2.9, respectively, which we employ in our argument to settle this subcase and complete the proof. □

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Declarations

Conflict of interest All the authors declare that there is no conflict of interest.

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