Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XXII (2021), 195-240

A weak notion of visibility, a family of examples, and Wolff–Denjoy theorems

GAUTAM BHARALI AND ANWOY MAITRA

Abstract. We investigate a form of visibility introduced recently by Bharali and Zimmer — and shown to be possessed by a class of domains called Goldilocks domains. The range of theorems established for these domains stems from this form of visibility together with certain quantitative estimates that define Goldilocks domains. We show that some of the theorems alluded to follow *merely* from the latter notion of visibility. We call those domains that possess this property visibility domains with respect to the Kobayashi distance. We provide a sufficient condition for a domain in $\mathbb{C}^n$ to be a visibility domain. A part of this paper is devoted to constructing a family of domains that are visibility domains with respect to the Kobayashi distance but are *not* Goldilocks domains. Our notion of visibility is reminiscent of uniform visibility in the context of CAT(0) spaces. However, this is an imperfect analogy because, given a bounded domain Ω in $\mathbb{C}^n$, $n \geqslant 2$, it is, in general, not even known whether the metric space $(\Omega, \mathsf{k}_\Omega)$ (where k_Ω is the Kobayashi distance) is a geodesic space. Yet, with just this weak property, we establish two new Wolff–Denjoy-type theorems.

Mathematics Subject Classification (2010): 32F45 (primary); 32H50, 53C23, 32U05 (secondary).

1. Introduction and statement of main results

This work is motivated by the results — ranging from the boundary behaviour of complex geodesics to the dynamics of iterations of holomorphic maps — in a recent work by Bharali and Zimmer [9]. In that work, the authors introduce a class of bounded domains in $\mathbb{C}^n$, called Goldilocks domains, and establish for these domains the range of results alluded to. Given any bounded domain $\Omega \subset \mathbb{C}^n$, let k_Ω be the Kobayashi distance on Ω and $\kappa_\Omega : \Omega \times \mathbb{C}^n \cong T^{1,0}\Omega \to [0, +\infty)$ be the infinitesimal Kobayashi metric (also called the Kobayashi–Royden metric).

Gautam Bharali is supported by a Swarnajayanti Fellowship (Grant No. DST/SJF/MSA-02/2013-14). Anwoy Maitra is supported by a scholarship from the Indian Institute of Science. Both authors are also supported by a UGC CAS-II grant (Grant No. F.510/25/CAS-II/2018(SAP-I)).

Received June 06, 2019; accepted in revised form October 02, 2019.
Published online March 2021.

Goldilocks domains are defined in terms of certain quantitative bounds from below on $\kappa_\Omega(z; \cdot)$ and from above on $\mathsf{k}_\Omega(o, z)$ (where o is some chosen point in Ω) as $z \to \partial\Omega$ — see Subsection 1.1 below for a precise definition. The results in [9] are a consequences of these bounds. In proving some of the major results in [9], these bounds play two separate roles:

(a) In controlling the oscillation of holomorphic maps, the magnitudes of their derivatives, etc., along sequences approaching $\partial\Omega$;
(b) In establishing that $(\Omega, \mathsf{k}_\Omega)$ has certain consequential features — first identified by Eberlein and O'Neill — possessed by manifolds with negative sectional curvature.

The property hinted at by (b) is a purely geometric (*i.e.*, not quantitative) property reminiscent of visiblity in the sense of Eberlein–O'Neill [14]. It is used in a fundamental way in the above-mentioned results. So, it is natural to ask whether the conclusions of those results would hold true in domains that **merely** have the geometric property alluded to — *i.e.*, without assuming the quantitative estimates that define Goldilocks domains. We coin a term for those domains that have this property via the following definition (see Subsection 1.2 for the definition of a (λ, κ)-almost-geodesic):

Definition 1.1. Let Ω be a bounded domain in $\mathbb{C}^n$. We say that Ω is a *visibility domain with respect to the Kobayashi distance* (or just *visibility domain* for brevity) if, given any $\lambda \geqslant 1$ and $\kappa \geqslant 0$, for each pair of distinct points $\xi, \eta \in \partial\Omega$ and each pair of $\overline{\Omega}$-open neighbourhoods V and W of ξ and η, respectively, such that $\overline{V} \cap \overline{W} = \varnothing$, there exists a compact subset K of Ω such that the image of each (λ, κ)-almost-geodesic $\sigma : [0, L] \to \Omega$ with $\sigma(0) \in V$ and $\sigma(L) \in W$ intersects K.

While the notion in the above definition is strongly reminiscent of the notion of visibility manifolds — especially in view of [5, pp. 54–55] — we must point out that the analogy is imperfect. For instance, given a bounded domain in $\mathbb{C}^n$, $n \geqslant 2$, it is, in general, not even known whether the metric space $(\Omega, \mathsf{k}_\Omega)$ is a geodesic space. It is for this reason that Definition 1.1 features (λ, κ)-almost-geodesics, which serve as substitutes for geodesics.

One might ask: is there a reasonably rich collection of domains that are visibility domains with respect to the Kobayashi distance? The answer to this is, "Yes," since any Goldilocks domain is a visibility domain with respect to the Kobayashi distance, and — as shown in [9] — the Goldilocks property admits a very wide range of domains. However, Definition 1.1 would be interesting only if one knew that there exist visibility domains that are **not** Goldilocks domains. A major part of this paper is devoted to showing that there is a rich family of domains of this sort. We introduce these domains in Subsection 1.1. In other words, Definition 1.1 is not just a geometrization of the Goldilocks property but also admits domains in $\mathbb{C}^n$ that are fundamentally different from Goldilocks domains.

Most consequences of visibility in the sense of [14] have been extended to CAT(0) spaces — see [10, Chapter II], for instance. Uniform visibility is the analogue, in the context of CAT(0) spaces, of the property given in Definition 1.1.

Now, a proper CAT(0) space is uniformly visible if and only if it is Gromov hyperbolic. There is a reason for mentioning this: many statements that one would like to prove for the metric space $(\Omega, \mathsf{k}_\Omega)$ would follow very easily if this space were Gromov hyperbolic. However, Gromov hyperbolicity is a property that is *extremely* difficult to establish for k_Ω for $\Omega \subset \mathbb{C}^n$ when $n \geqslant 2$ — see [6,28] for some positive instances. Visibility, in the sense of Definition 1.1, is much easier to show. In Theorem 1.5 below we present fairly mild conditions for a bounded domain in $\mathbb{C}^n$ to be a visibility domain. It is this theorem that we use to show that the domains introduced in Subsection 1.1 are visibility domains with respect to the Kobayashi distance. We expect that their construction would serve as a general recipe for constructing visibility domains.

Returning to the question in our first paragraph: the link between Gromov hyperbolicity and the property in Definition 1.1, via analogies to uniform visibility, continues to motivate (as in the case of [9]) certain key moves in proving analogues of some of the results in [9]. But we show here that the roles of the quantitative bounds (which also define Goldilocks domains) identified in (a) above can often be managed by the visibility property alone. This is the content of our results in Section 4, which may be of independent interest. With these inputs, we can systematically approach several applications for which visibility is well-suited — some of which will be a part of forthcoming work. In this paper, we prove two Wolff–Denjoy-type theorems, which we introduce in Subsection 1.3.

We now introduce the main theorems of this paper.

1.1. Visibility domains that are not Goldilocks domains

We begin with the definition of a Goldilocks domain. For this, we shall need two quantities. Given a bounded domain Ω and a point $z \in \Omega$, $\delta_\Omega(z)$ will denote the (Euclidean) distance from z to $\mathbb{C}^n \setminus \Omega$. Next, we define:

$$M_\Omega(r) := \sup\left\{\frac{1}{\kappa_\Omega(z; v)} \mid \delta_\Omega(z) \leqslant r \text{ and } \|v\| = 1\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm (the choice of a norm is actually irrelevant to the purpose that M_Ω serves). From the definition of κ_Ω, it is easy to see that M_Ω expresses the lower bound for κ_Ω on the unit sphere in $T^{1,0}\Omega$ in terms of the distance from $\mathbb{C}^n \setminus \Omega$.

Definition 1.2. A bounded domain $\Omega \subset \mathbb{C}^n$ is called a *Goldilocks domain* if

(1) for some (hence any) $\epsilon > 0$ we have

$$\int_0^\epsilon \frac{1}{r} M_\Omega(r)\, dr < \infty;$$

(2) for each $z_0 \in \Omega$ there exist constants $C, \alpha > 0$ (that depend on z_0) such that

$$\mathsf{k}_\Omega(z_0, z) \leqslant C + \alpha \log \frac{1}{\delta_\Omega(z)} \quad \forall z \in \Omega. \tag{1.1}$$

The quantitative bounds in the above definition encode the following idea: in a Goldilocks domain, $\kappa_\Omega(z;\ \boldsymbol{\cdot})$ cannot grow too slowly and $\mathsf{k}_\Omega(z_0, z)$ cannot grow too rapidly as $z \to \partial\Omega$ (this is the rationale for the term "Goldilocks domains").

The latter has the following geometric implication: if Ω is a Goldilocks domain, then $\partial\Omega$ can neither have outward-pointing cusps nor points at which $\partial\Omega$ is flat to infinite order and is, in a precise sense, too flat. One may intuit the assertion about outward-pointing cusps with just a little work: a classical argument for planar domains reveals that Condition 2 above fails for such domains. This is the intuition behind a family of domains — which we call *caltrops* — that are **not** Goldilocks domains, but whose geometry is sufficiently well-behaved that it is reasonable to expect them to be visibility domains. With this, we make the following:

Definition 1.3. A bounded domain $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, is called a *caltrop* if there exists a finite set of exceptional points $\{q_1, \dots, q_N\} \subset \partial\Omega$ such that $\partial\Omega \setminus \{q_1, \dots, q_N\}$ is $\mathcal{C}^2$-smooth, if $\partial\Omega$ is strongly Levi-pseudoconvex at each point in $\partial\Omega\setminus\{q_1,\dots,q_N\}$, and if for each exceptional point q_j, $j = 1, \dots N$, there exists a connected open neighbourhood $V_j \ni q_j$ such that $\Omega \cap V_j$ is described as follows: there exist constants $p_j \in (1, 3/2)$ and $C_j > 1$, a unitary transformation $\mathsf{U}^{(j)}$, and a continuous function $\psi_j : [0, A_j] \to [0, +\infty)$ (where $A_j > 0$) with the properties mentioned below such that $\mathsf{U}_j(\Omega \cap V_j)$ is a "solid of revolution" given by

$$\begin{aligned}&\mathsf{U}_j(\Omega \cap V_j)\\ &= \Big\{(z_1,\dots,z_n)\in\mathbb{C}^n \mid \Re(z_n)\in(0,A_j),\ \Im(z_n)^2+\textstyle\sum_{1\leqslant j\leqslant(n-1)}|z_j|^2<\psi_j\big(\Re(z_n)\big)^2\Big\},\end{aligned}$$

where we write $\mathsf{U}_j := \mathsf{U}^{(j)}(\boldsymbol{\cdot} - q_j)$. Each function ψ_j has the following properties:

- ψ_j is of class $\mathcal{C}^2$ on $(0, A_j)$;
- For each $x \in [0, A_j]$, we have
$$(1/C_j)\, x^{p_j} \leqslant \psi_j(x) \leqslant C_j\, x^{p_j};$$
- ψ_j is strictly increasing and ψ_j' is increasing on $(0, A_j)$;
- $\lim_{x\to 0^+} \psi_j(x)\psi_j''(x) = 0$.

A few words about the functions ψ_j in the above definition — and about the last (somewhat technical-looking) property — are in order. These functions are meant to quantify the fact that, around each point of $\partial\Omega$ at which it is non-smooth, the boundary resembles the following real (singular) hypersurface

$$\Big\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \Re(z_n) \in (0, A),\ \Im(z_n)^2 + \textstyle\sum_{1\leqslant j\leqslant(n-1)}|z_j|^2 = \Re(z_n)^{2p}\Big\}$$

for some $p \in (1, 3/2)$. The latter models a Hölderian cusp that is not too sharp.

The reader may wonder, given that several specific properties must hold true simultaneously in a caltrop, whether such a domain as described in Definition 1.3 can even exist. We show in Section 3 that caltrops do exist. With this, the assertion about caltrops that is of greatest interest to us is:

Theorem 1.4. *Caltrops are visibility domains with respect to the Kobayashi distance. However, a caltrop is* not *a Goldilocks domain.*

The proof of this theorem requires several supporting results — about which we shall say more presently — plus a sufficient condition for a bounded domain to be a visibility domain with respect to the Kobayashi distance. We discuss this sufficient condition next.

1.2. A sufficient condition for visibility

The following is the sufficient condition that we have alluded to several times in this section.

Theorem 1.5 (General visibility lemma). *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Suppose there exists a $\mathcal{C}^1$-smooth strictly increasing function $f : (0, +\infty) \to \mathbb{R}$ such that*

- *$f(t) \to +\infty$ as $t \to +\infty$;*
- *For some z_0, we have*

$$\mathsf{k}_\Omega(z_0, z) \leqslant f\Big(\frac{1}{\delta_\Omega(z)}\Big) \quad \forall z \in \Omega.$$

Assume that $M_\Omega(t) \to 0$ as $t \to 0$ and that there exists an $r_0 > 0$ such that

$$\int_0^{r_0} \frac{M_\Omega(r)}{r^2}\, f'\Big(\frac{1}{r}\Big)\, dr < \infty. \tag{1.2}$$

Then, Ω is a visibility domain with respect to the Kobayashi distance.

It is clear – comparing Theorem 1.5 with Conditions (1) and (2) in Definition 1.2 – that our result is influenced by the definition of Goldilocks domains. Among our motivations were:

- that our conditions account for the estimates on k_Ω and κ_Ω when Ω is any of the planar domains referred to in Subsection 1.1 with $\partial\Omega$ having outward-pointing cusps (which also play a central role in establishing that caltrops are visibility domains);
- that elements of the proof of the main visibility result in [9], namely: [9, Theorem 1.4], continue to be useful in establishing visibility (in the sense of Definition 1.1).

- Observe that the inequalities that define Goldilocks domains are subsumed by Theorem 1.5: for these domains, just set

$$f(t) = C + \alpha \log(t), \quad t \in (0, +\infty),$$

with $C, \alpha > 0$ as in (1.1), in the latter theorem. The proof of Theorem 1.5 is given in Section 5.

There are two essential matters relating to visibility domains that we had deferred. We address them here. First, we give a definition for (λ, κ)-almost-geodesics:

Definition 1.6 (Bharali–Zimmer, [9]). Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and $I \subset \mathbb{R}$ an interval. For $\lambda \geqslant 1$ and $\kappa \geqslant 0$, a curve $\sigma : I \to \Omega$ is called a (λ, κ)*-almost-geodesic* if

(1) for all $s, t \in I$,

$$\lambda^{-1}|t - s| - \kappa \leqslant \mathsf{k}_\Omega(\sigma(s), \sigma(t)) \leqslant \lambda|t - s| + \kappa;$$

(2) σ is absolutely continuous (whence $\sigma'(t)$ exists for almost every $t \in I$) and, for almost every $t \in I$, $\kappa_\Omega(\sigma(t); \sigma'(t)) \leqslant \lambda$.

Secondly, the discussion surrounding visibility domains suggests that, given any pair of points in Ω, there exists (for any $\lambda \geqslant 1$ and $\kappa > 0$) a (λ, κ)-almost-geodesic joining them, but why must this be so? In fact, this is true for **any** bounded domain Ω — as shown by [9, Proposition 4.4].

The General Visibility Lemma (*i.e.*, Theorem 1.5) is our key tool for showing that caltrops are visibility domains. We must derive appropriate upper bounds for $\mathsf{k}_\Omega(o, z)$ (where o is a chosen point in Ω) and lower bounds for $\kappa_\Omega(z;\ \boldsymbol{\cdot})$ for z in the caltrop Ω, z sufficiently close to $\partial\Omega$. The following points describe very briefly the challenging parts of the proof of Theorem 1.4:

(i) We explicitly calculate the Kobayashi distance on a model domain $D \Subset \mathbb{C}$, ∂D having an outward-pointing cusp, which is carefully chosen keeping in view the geometry of $\partial\Omega \cap V_j$, $j = 1, \dots, N$, the latter being an outward-pointing cusp of $\partial\Omega$ as described in Definition 1.3. Then, $\mathbb{C}$-affine embeddings of D into $\Omega \cap V_j$ allow us to estimate k_Ω using the fact that these embeddings are contractive relative to the Kobayashi distance.

(ii) We introduce a trick of estimating the Sibony pseudometric [25, Proposition 6] for Ω in $\Omega \cap V_j$, $j = 1, \dots, N$. The relationship between the latter pseudometric and κ_Ω leads to an estimate for $\kappa_\Omega(z;\ \boldsymbol{\cdot})$ from below for $z \in \Omega \cap V_j$.

The argument summarized by (i) above requires several results, which are presented in Section 6. The proof of Theorem 1.4 is given in Section 7. We expect the procedures in this proof to serve as a general recipe for constructing new visibility domains.

1.3. Wolff–Denjoy theorems for visibility domains

The classical Wolff–Denjoy theorem is as follows (in this paper, $\mathbb{D}$ will denote the open unit disk in $\mathbb{C}$ with centre 0):

Result 1.7 (Denjoy [13], Wolff [26]). Suppose $F : \mathbb{D} \to \mathbb{D}$ is a holomorphic map. Then, one of the following holds:

(1) F has a fixed point in $\mathbb{D}$;
(2) There exists a point $\xi \in \partial\mathbb{D}$ such that $\lim_{\nu\to\infty} F^{\nu}(z) = \xi$ for every $z \in \mathbb{D}$, this convergence being uniform on compact subsets of $\mathbb{D}$.

There has been sustained interest in understanding the behaviour of iterates of a map $F : X \to X$ where X is a space that — in some appropriate sense — resembles $\mathbb{D}$ while F possesses some degree of regularity that enables a generalization of Result 1.7 to $F : X \to X$. The above result was extended to the unit (Euclidean) ball in $\mathbb{C}^n$, for all $n \in \mathbb{Z}_+$, by Hervé [16]. Abate further generalized this in [1] to strongly convex domains. Let X be a visibility manifold in the sense of [14] (as discussed at the beginning of this section). Then, one can construct a boundary for X "at infinity" that serves as the analogue of $\partial\mathbb{D}$. With this set-up for X, Beardon [7] generalized the above result to $F : X \to X$ where F is a strict contraction.

For various reasons having to do with their intrinsic geometry, convex domains predominate among recent generalizations of the Wolff–Denjoy theorem: see, for instance, [4, 8, 11, 29] and several of the results in [20]. Visibility in the sense of Definition 1.1 is one of the key ingredients in the proof by Bharali–Zimmer of a generalization [9, Theorem 1.10] of Result 1.7 to taut Goldilocks domains. This extends the Wolff–Denjoy phenomenon to a wide range of domains, including pseudoconvex domains of finite type — see [9, Corollary 2.11] — which are, in general, neither convex nor biholomorphic to convex domains. In the following result, we extend the Wolff–Denjoy phenomenon to **all** visibility domains with respect to the Kobayashi distance that are taut:

Theorem 1.8. *Suppose $\Omega \subset \mathbb{C}^n$ is a visibility domain with respect to the Kobayashi distance that is taut and that $F : \Omega \to \Omega$ is a holomorphic map. Then exactly one of the following holds:*

(1) *For each $z \in \Omega$, the orbit $\{F^{\nu}(z) \mid \nu \in \mathbb{Z}_+\}$ is relatively compact in Ω;*
(2) *There exists a $\xi \in \partial\Omega$ such that $\lim_{\nu\to\infty} F^{\nu}(z) = \xi$ for every $z \in \Omega$, this convergence being uniform on compact subsets of Ω.*

The statement of the above theorem differs from that of [9, Theorem 1.10] in the one respect that the dichotomy presented in the above theorem holds true on any taut visibility domain, and not just on Goldilocks domains. Furthermore, the hypothesis of Theorem 1.8 does admit domains that are not Goldilocks domains — as the reader will infer from Corollary 1.10 below.

The proof of [9, Theorem 1.10] is borne by two distinct ideas. The first is the following heuristic, which is entirely a consequence of visibility: if we assume that there exist two strictly increasing sequences $(\nu_i)_{i\geqslant1}, (\mu_j)_{j\geqslant1} \subset \mathbb{Z}_+$ with $F^{\nu_i}(o) \to \xi \in \partial\Omega$, $F^{\mu_j}(o) \to \eta \in \partial\Omega$, and $\xi \neq \eta$, then we arrive at a contradiction by analysing $\mathsf{k}_\Omega(F^{\nu_i}(o), F^{\mu_j}(o))$. Briefly: with $\nu_i > \mu_j$, an estimate for $\mathsf{k}_\Omega(F^{\nu_i}(o), F^{\mu_j}(o))$ based on the fact that F is contractive with respect to k_Ω is incompatible with an estimate based on the fact that every $(1, \kappa)$-almost-geodesic (with $\kappa > 0$) joining $F^{\nu_i}(o)$ to $F^{\mu_j}(o)$ must pass within a fixed distance of o. As this is a consequence of visibility, this heurisitc informs the proof of Theorem 1.8 too. **However,** visibility alone does not a priori seem to explain the limits $F^{\nu_i}(o) \to \xi$ and $F^{\mu_j}(o) \to \eta$ mentioned above. That explanation, under the assumption that Ω is taut, belongs to the realm described by (a) earlier in this section. It turns out that (still assuming that Ω is taut) visibility alone is enough to justify these limits. This is the purport of our results in Section 4, which play a supporting role in the proof of Theorem 1.8, but may also be of independent interest.

The dichotomy in the behaviour of the iterations in Theorem 1.8 is not quite what is given by the classical Wolff–Denjoy theorem (*i.e.*, Result 1.7). Where, among the Wolff–Denjoy-type results cited above, the dichotomy given by Result 1.7 does hold, it is a consequence of the domains Ω in question being contractible and of $\partial\Omega$ satisfying some non-degeneracy condition: some form of strict convexity; or strong pseudoconvexity, as in [18]; etc. In view of the many examples presented in [9, Section 2], and given Theorem 1.4 about caltrops, the boundaries of taut visibility domains do not generally have the type of non-degeneracy mentioned above. However, with some conditions on the topology of Ω (as opposed to the geometry of $\partial\Omega$), we can use a result of Abate [3] to obtain a version of Theorem 1.8 whose conclusions more closely resemble those of Result 1.7.

Theorem 1.9. *Suppose $\Omega \subset \mathbb{C}^n$ is a visibility domain with respect to the Kobayashi distance that is taut, and that Ω is of finite topological type. Suppose further that*

$$H^j(\Omega; \mathbb{C}) = 0 \quad \text{for each odd } j,\ 1 \leqslant j \leqslant n.$$

Let $F : \Omega \to \Omega$ be a holomorphic map. Then exactly one of the following holds:

(1) *F has a periodic point in Ω;*
(2) *There exists a $\xi \in \partial\Omega$ such that $\lim_{\nu\to\infty} F^\nu(z) = \xi$ for every $z \in \Omega$, this convergence being uniform on compact subsets of Ω.*

In the first case, each orbit $\{F^\nu(z) \mid \nu \in \mathbb{Z}_+\}$ for $z \in \Omega$, is relatively compact in Ω.

Recall that for Ω to have finite topological type means that the singular homology groups $H_j(\Omega; \mathbb{Z})$ are of finite rank for all $j \in \mathbb{N}$.

We point out that the domains to which Theorem 1.9 applies need *not* be convex or biholomorphic to a convex domain (a fact that will be emphasised by the corollary below). In this regard, Theorem 1.9 bears relation to [18] by X. Huang, in which the dichotomy presented in Result 1.7 is established for bounded topologically contractible strongly pseudoconvex domains. Loosely speaking, a version of the heuristic discussed right after Theorem 1.8 appears in [18], although the specifics that make this heuristic work in [18] and for our result differ greatly. (Also, the arguments in [18] suggest that the dichotomy presented in Result 1.7 would be very hard to obtain for the domains of the generality that we consider.)

The final result of this subsection is meant to illustrate tangibly the range of domains — with an emphasis on domains that need not be convex or biholomorphic to a convex domain, and with boundaries that aren't even Lipschitz — to which the Wolff–Denjoy phenomenon extends.

Corollary 1.10. *Suppose $\Omega \subset \mathbb{C}^n$ is either a bounded pseudoconvex domain of finite type or a caltrop. Suppose further that Ω is of finite topological type and that*

$$H^j(\Omega; \mathbb{C}) = 0 \quad \text{for each odd } j,\ 1 \leqslant j \leqslant n.$$

Let $F : \Omega \to \Omega$ be a holomorphic map. Then exactly one of the following holds:

(1) *F has a periodic point in Ω;*
(2) *There exists a $\xi \in \partial\Omega$ such that $\lim_{\nu\to\infty} F^\nu(z) = \xi$ for every $z \in \Omega$, this convergence being uniform on compact subsets of Ω.*

In the first case, each orbit $\{F^\nu(z) \mid \nu \in \mathbb{Z}_+\}$,for $z \in \Omega$, is relatively compact in Ω.

The proofs of all the results of this subsection are presented in Section 9 below.

2. Technical preliminaries

This section is dedicated to introducing notation that will recur throughout this paper, and some known results that will play a supporting role in the proofs presented in the following sections. This section is divided into three parts. We begin with some notation (some of which has appeared in passing in Section 1) that we shall need.

2.1. Common notation

We fix the following notation, which we shall frequently need.

(1) For $v \in \mathbb{C}^n$, $\|v\|$ will denote the Euclidean norm. Given points $z, w \in \mathbb{C}^n$, we shall commit a mild abuse of notation by not distinguishing between points and tangent vectors, and denote the Euclidean distance between them as $\|z - w\|$.
(2) The maps $\pi_j : \mathbb{C}^n \to \mathbb{C}$, $j = 1, \dots, n$, will denote the projection onto the j-th factor.
(3) $\mathbb{D}$ will denote the open unit disk in $\mathbb{C}$ with centre at 0, while $D(a, r)$ will denote the open disk in $\mathbb{C}$ with radius $r > 0$ and centre a.
(4) Given an open set $U \subset \mathbb{C}^n$ and a $\mathcal{C}^2$-smooth function $\rho : U \to \mathbb{R}$, we will denote by $\mathscr{L}(\rho)(z; v)$ the quadratic form (called the *Levi-form* of ρ) determined by the complex Hessian of ρ at $z \in U$:

$$\mathscr{L}(\rho)(z; v) := \sum_{1\leqslant j,\, k\leqslant n} \partial^2_{z_j\bar{z}_k}\rho(z) v_j \overline{v}_k$$

for each $v \in \mathbb{C}^n$ (equivalently, for each $v \in T^{1,0}_z U$).

2.2. Facts relating to the Kobayashi geometry of domains

Let Ω be a domain in $\mathbb{C}^n$. We shall assume that the reader is familiar with the Kobayashi pseudodistance k_Ω and the Kobayashi–Royden pseudometric κ_Ω. The only comment concerning the basics of these objects that we shall make is that k_Ω and κ_Ω are related as follows:

$$\mathsf{k}_\Omega(z, w) = \inf_{\gamma \in \mathscr{C}(z,w)} \int_0^1 \kappa_\Omega(\gamma(t); \gamma'(t))\, dt \quad \forall z, w \in \Omega,$$

where $\mathscr{C}(z, w)$ is the set of all piecewise $\mathcal{C}^1$ paths $\gamma : [0, 1] \to \Omega$ satisfying $\gamma(0) = z$ and $\gamma(1) = w$. This is a result by Royden [24]

We shall need the following estimate on k_Ω:

Result 2.1 ([9, Proposition 3.5-(1)]). Let Ω be a bounded domain in $\mathbb{C}^n$. Fix an open ball $\mathbb{B}(\Omega)$ with centre $0 \in \mathbb{C}^n$ that is so large that $\Omega \Subset \mathbb{B}(\Omega)$. Let

$$c := \inf_{x \in \overline{\Omega},\ \|v\|=1} \kappa_{\mathbb{B}(\Omega)}(x; v).$$

Then, $\mathsf{k}_\Omega(z, w) \geqslant c\|z - w\|$ for every $z, w \in \Omega$.

Tautness is closely tied to metric geometry associated to the Kobayashi pseudodistance. For a domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, being taut provides additional information about the complex geometry of Ω. We collect a couple of observations of this nature in the following:

Result 2.2. Let $\Omega \subset \mathbb{C}^n$ be a taut domain. Then:

(1) (see, for instance, [19, Proposition 3.5.13]) κ_Ω is continuous on $\Omega \times \mathbb{C}^n$;
(2) ([27, Theorem F]) If Ω is bounded, then it is pseudoconvex.

In order to prove Corollary 1.10, we would need to prove that caltrops (we shall show in the next section that caltrops indeed exist) are taut. The following result will be useful in this proof.

Result 2.3 ([15, Corollary 2.4]). Let Ω be a bounded domain in $\mathbb{C}^n$ whose boundary is of class $\mathcal{C}^2$ and strongly Levi-pseudoconvex in $\partial\Omega$-neighbourhoods of two distinct points $\xi, \eta \in \partial\Omega$. Then, there is a constant $C > 0$, which depends on ξ and η, and open neighbourhoods V_ξ and V_η in $\mathbb{C}^n$ of ξ and η, respectively such that

$$\mathsf{k}_\Omega(a, b) \geqslant 2^{-1} \log \frac{1}{\delta_\Omega(a)} + 2^{-1} \log \frac{1}{\delta_\Omega(b)} - C$$

for each point $a \in \Omega \cap V_\xi$ and $b \in \Omega \cap V_\eta$.

The proof of Theorem 1.4 will, at a certain stage, require a precise estimate from below for κ_Ω — where Ω is a bounded domain in $\mathbb{C}^n$ — in the vicinity of a strictly pseudoconvex point in $\partial\Omega$. Such an estimate is provided by a result of D. Ma [23, Theorem B]. Before stating the result that we need, we ought to mention that Ma's result is stated for domains for which the part of the boundary that is strongly Levi-pseudoconvex is $\mathcal{C}^3$-smooth. However, Ma's techniques are still valid *up to a point* when this regularity condition is weakened to $\mathcal{C}^2$-smooth — the modifications required, in essence, are to replace all occurences of $O(\|x\|^3)$ by $o(\|x\|^2)$ in those steps of the argument that invoke Taylor's theorem. Where this does not suffice, Balogh and Bonk — in the sketch of their proof of [6, Proposition 1.2] — provide the essential modification needed. While Balogh–Bonk state their estimate for strongly pseudoconvex domains with $\mathcal{C}^2$-smooth boundary, their proof actually involves *local* estimates, which lead to the inequalities below. With these clarifications, we state:

Result 2.4 (paraphrasing [23, Theorem B] and [6, Proposition 1.2]). Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geqslant 2$. Let $\mathcal{M}_0$ be a $\partial\Omega$-open set that is a $\mathcal{C}^2$-smooth hypersurface. Assume that $\mathcal{M}_0$ admits a defining function ϕ that is of class $\mathcal{C}^2$ on some open set containing $\mathcal{M}_0$ and that there exists a small constant $\sigma > 0$ such that $\mathscr{L}(\phi)(\xi; v) \geqslant \sigma\|v\|^2$ at each $\xi \in \mathcal{M}_0$ and for all $v \in \mathbb{C}^n$. Let $\mathcal{M}_1 \subsetneq \mathcal{M}_0$ be a compact subset. Then, there exists an $\overline{\Omega}$-open neighbourhood, say $\mathcal{V}$, of $\mathcal{M}_1$ and a constant $C > 0$ such that

$$\kappa_\Omega(z; v) \geqslant (1 - C\delta_\Omega(z)^{1/2}) \frac{\sigma\|v\|^2}{\delta_\Omega(z)^{1/2}}$$

for every $z \in \mathcal{V} \cap \Omega$ and for every $v \in \mathbb{C}^n$.

Remark 2.5. In fact, a much more precise estimate is provided by Ma and Balogh–Bonk than the one stated in the above result. However, in order to state the latter estimate, one would need to provide certain definitions that would be a digression from the present discussion. The lower bound for κ_Ω stated in Result 2.4 suffices for our purposes.

The following result by Sibony will also play an important role in the proof of Theorem 1.4:

Result 2.6 (paraphrasing [25, Proposition 6]). Let Ω be a domain in $\mathbb{C}^n$ and let $p \in \Omega$. Suppose u is a negative plurisubharmonic function that is of class $\mathcal{C}^2$ in a neighbourhood of p and assume that

$$\mathscr{L}(u)(p; v) \geqslant c\|v\|^2 \quad \forall v \in \mathbb{C}^n,$$

where c is some positive constant. Then, there is a universal constant $\alpha > 0$ such that

$$\kappa_\Omega(p; v) \geqslant \left(\frac{c}{\alpha}\right)^{1/2} \frac{\|v\|}{|u(p)|^{1/2}}.$$

Remark 2.7. An important part of [25] is the construction of a pseudometric on $T^{1,0}\Omega$ — which is known today as the Sibony pseudometric — that is dominated by the Kobayashi pseudometric. The lower bound in Result 2.6 is actually a lower bound for the Sibony pseudometric, from which the lower bound above for $\kappa_\Omega(p; \cdot)$ is obtained.

In concluding this section, we collect a few

2.3. Facts relating to length-minimizing curves

The fundamental fact that we presuppose in this subsection is that if Ω is a bounded domain in $\mathbb{C}^n$, then for any two points in Ω and for any $\lambda \geqslant 1$ and $\kappa > 0$ there exists a (λ, κ)-almost-geodesic joining these points: this is the content of Proposition 4.4 of [9] by Bharali–Zimmer. With this understanding, we first present:

Result 2.8 ([9, Proposition 4.3]). Let Ω be a bounded domain in $\mathbb{C}^n$. For any $\lambda \geqslant 1$ there exists a $C = C(\lambda) > 0$ such that any (λ, κ)-almost-geodesic (where $\kappa \geqslant 0$) $\sigma : [a, b] \to \Omega$ is C-Lipschitz (with respect to the Euclidean distance).

We shall also need the following simple lemma, whose proof is essentially a single line following from the definition of (λ, κ)-quasi-geodesics and the triangle inequality. To clarify: a (λ, κ)-*quasi-geodesic* in Ω is a function $\sigma : I \to \Omega$, where I is an interval, satisfying the property (1.6) stated in Definition 1.6.

Lemma 2.9. *Let Ω be a bounded domain in $\mathbb{C}^n$. If $\sigma : [a, b] \to \Omega$ is a $(1, \kappa)$-quasi-geodesic, then for all $t \in [a, b]$ we have*

$$\mathsf{k}_\Omega(\sigma(a), \sigma(b)) \leqslant \mathsf{k}_\Omega(\sigma(a), \sigma(t)) + \mathsf{k}_\Omega(\sigma(t), \sigma(b)) \leqslant \mathsf{k}_\Omega(\sigma(a), \sigma(b)) + 3\kappa.$$

3. Caltrops exist

In this section, we shall construct explicit examples of caltrops. To begin with, we will construct with some care a caltrop whose boundary has one outward-pointing cusp. We shall then abstract features of this construction to describe briefly the construction of caltrops with any (finite) number of outward-pointing cusps. Our constructions will be in $\mathbb{C}^2$ but — as will become clear — this is only for simplicity of notation.

We shall call the subset $\Omega \cap V_j$, $j = 1, \dots, N$, where V_j is as in Definition 1.3, a *spike*.

3.1. A caltrop with a single spike

Let A and β be positive numbers and let $\psi : [-A, \beta] \to [0, +\infty)$ be a continuous function that is of class $\mathcal{C}^2$ on $(-A, \beta)$ such that

(1) $\psi(t) := (t + A)^p$ for every $t \in [-A, -B]$;
(2) $\psi(t) := \sqrt{\beta^2 - t^2}$ for every $t \in (0, \beta)$,

where $B \in (0, A)$ and $p \in (1, 3/2)$. We shall consider the following "solid of revolution" given by

$$\Omega := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |\Im w|^2 < C\psi(\Re w)^2,\ -A < \Re(w) < \beta\},$$

where $C > 0$ is a small constant whose magnitude we shall specify presently. Let us write

$$\rho(z, w) := |z|^2 + |\Im w|^2 - C\psi(\Re w)^2, \quad (z, w) \in \{(z, w) \in \mathbb{C}^2 : -A < \Re(w) < \beta\}.$$

It is easy to check that ρ is a $\mathcal{C}^2$-smooth defining function for the real hypersurface $\partial\Omega \cap \{(z,w) \in \mathbb{C}^2 : -A < \Re(w) < \beta\}$.

We compute:

$$\begin{aligned} \partial^2_{z\bar{z}}\rho &\equiv 1, \\ \partial^2_{z\overline{w}}\rho = \partial^2_{\bar{z}w}\rho &\equiv 0, \\ \partial^2_{w\overline{w}}\rho(z,w) &= \frac{1}{2} - \frac{C}{2}\left(\psi''(\Re w)\psi(\Re w) + \psi'(\Re w)^2\right), \end{aligned}$$

wherever ρ is of class $\mathcal{C}^2$. In particular, we have

$$\partial^2_{w\overline{w}}\rho(z,w) - \frac{1}{2} = -\frac{Cp(2p-1)}{2}(\Re w + A)^{2(p-1)} \nearrow 0 \text{ as } \Re w \searrow -A. \quad (3.1)$$

Furthermore, as ψ is of class $\mathcal{C}^2$ on $(-A, \beta)$, we can, by choosing $C > 0$ sufficiently small, ensure that

$$\partial^2_{w\overline{w}}\rho(z,w) \geqslant \frac{1}{4} \quad \forall w : -A < \Re w \leqslant 0. \quad (3.2)$$

From (3.1) and (3.2), we conclude that $\rho|_{\{(z,w)\in\mathbb{C}^2 : -A<\Re w<\delta\}}$ is strictly plurisubharmonic for some positive constant $\delta \ll 1$. In particular, $\partial\Omega$ is strongly Levi-pseudoconvex at each point on $\partial\Omega \cap \{(z,w) \in \mathbb{C}^2 : -A < \Re w \leqslant 0\}$. Of course, by construction — by the condition (2) on ψ, to be precise — $\partial\Omega$ is strongly Levi-pseudoconvex at each point on $\partial\Omega \cap \{(z,w) \in \mathbb{C}^2 : \Re w \geqslant 0\}$. The other properties that Ω must have for it to be a caltrop follow from the condition (1) on ψ.

3.2. A caltrop with many spikes

A slight modification of the details described in the previous subsection allows us to show the existence of caltrops with many spikes. To this end, let us fix constants $A_1, \dots A_N > 1$ and consider a collection of continuous functions $\psi_j : [-A_j, \beta_j] \to [0, +\infty)$ — where each β_j is a constant with $\beta_j > -1$ — that are of class $\mathcal{C}^2$ on $(-A_j, \beta_j)$, such that

$$\psi_j(t) := (t + A_j)^{p_j} \quad \forall t \in [-A_j, -B_j],$$

and where $B_j \in (1, A_j)$, $p_j \in (1, 3/2)$, $j = 1, \dots, N$. The precise values of the constants β_j and the properties of each $\psi_j\big|_{-B_j<t\leqslant\beta_j}$, $j = 1, \dots, N$, are determined by the construction that follows. Consider the "hypersurfaces of revolution"

$$\mathscr{H}_j := \{(z,w) \in \mathbb{C}^2 : |z|^2 + |\Im w|^2 = C_j\psi_j(\Re w)^2,\ -A_j \leqslant \Re(w) \leqslant \beta_j\},$$

where each C_j is a positive constant whose value we shall fix appropriately. Now, consider the Euclidean unit sphere $S^3 \subset \mathbb{C}^2$ and pick N distinct points $p_1, \dots, p_N \in$

S^3, $N \geqslant 2$. Fix unitary transformations U_j (relative to the standard Hermitian inner product on $\mathbb{C}^2$) such that

$$\mathsf{U}_j(0, -1) = p_j$$

for each $j = 1, \dots, N$. Now consider the half-spaces $\Sigma_j := \{(z, w) \in \mathbb{C}^2 : \Re(w) < -1 + \delta_j\}$, where each $\delta_j > 0$ is a *small* constant. Let

$$\mathsf{C}_j := S^3 \cap \mathsf{U}_j(\Sigma_j),$$

$j = 1, \dots, N$; these are small caps on the sphere. It follows from the discussion in the previous subsection that, by adjusting the values of the constants B_j, β_j, C_j and δ_j introduced above appropriately, we can define each ψ_j, $j = 1, \dots, N$, in such a way that the set

$$\mathcal{S} := \left(S^3 \setminus \cup_{j=1}^N \mathsf{C}_j\right) \bigcup \left(\cup_{j=1}^N \mathsf{U}_j(\mathscr{H}_j)\right)$$

is a compact topological submanifold such that $\mathcal{S} \setminus \{q_1, \dots, q_N\}$ — where $q_j := \mathsf{U}_j(0, -A_j)$, $j = 1, \dots, N$ — is of class $\mathcal{C}^2$ and is strongly Levi-pseudoconvex at each of its points. It is then easy to show that the bounded component of $\mathbb{C}^2 \setminus \mathcal{S}$ is a caltrop.

The next subsection is, strictly speaking, unrelated to the issue of the existence of caltrops. But, having shown that caltrops in $\mathbb{C}^2$ exist, it is easy to see that the construction above can be generalized to $\mathbb{C}^n$ for every $n \geqslant 2$. We would like to extend the Levi-form calculation in Subsection 3.1 to higher dimensions. This will be needed in the proof of Theorem 1.4. Thus, we conclude this section with the following:

3.3. A Levi-form calculation for caltrops

We would like — in proving Theorem 1.4 — to observe the notation introduced in Definition 1.3. Thus, we present the following lemma that follows a calculation analogous to the one in Subsection 3.1.

Lemma 3.1. *Let $\Omega \subset \mathbb{C}^n$, $n \geqslant 2$, be a caltrop. Let q denote one of the points $\{q_1, \dots, q_N\} \subset \partial\Omega$ (say q_{j^*}) as in Definition* 1.3 — i.e., *one of the tips of a spike of Ω. Let $\psi : [0, A] \to [0, +\infty)$, $V \ni q$ and $p \in (1, 3/2)$ denote the data associated to q by Definition* 1.3. *Let $(z_1, \dots, z_n)$ represent the system of global holomorphic coordinates centered at $q\,(= q_{j^*})$ obtained by the transformation of the product coordinates on $\mathbb{C}^n$ by the map U_{j^*}. Let us abbreviate $(z_1, \dots, z_{n-1}, z_n)$ as (z', z_n). Then,*

(1) *The function*

$$\rho(z) := \Im(z_n)^2 + \|z'\|^2 - \psi(\Re(z_n))^2, \ \ (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}, \ z_n : 0 < \Re(z_n) < A,$$

is a defining function of $\mathsf{U}_{j^*}(\partial\Omega) \cap \{(z_1, \dots, z_n) \mid 0 < \Re(z_n) < A\}$;

(2) *The Levi-form of ρ is given by*

$$\begin{aligned}\mathscr{L}(\rho)(z;v) &= \|(v_1,\dots,v_{n-1})\|^2\\ &\quad+\left(\frac{1}{2}-\frac{1}{2}\big(\psi''(\Re z_n)\psi(\Re z_n)+\psi'(\Re z_n)^2\big)\right)|v_n|^2\\ &\quad\forall z\in \mathsf{U}_{j^*}(\partial\Omega)\cap\{(z_1,\dots,z_n)\mid 0<\Re(z_n)<A\} \text{ and } \forall v\in\mathbb{C}^n.\end{aligned}$$

Proof. A simple calculation reveals that $d\rho(z)\neq 0$ at each $z\in\mathsf{U}_{j^*}(\partial\Omega)\cap\{(z_1,\dots,z_n)\mid 0<\Re(z_n)<A\}$. This, together with the explicit description of each spike of Ω, establishes that ρ is a (local) defining function for the stated piece of $\partial\Omega$.

The calculations required for determinining the Levi-form are analogous to those in Subsection 3.1. Specifically:

$$\begin{aligned}\partial^2_{z_j\bar z_j}\rho &\equiv 1 \quad\text{for } j=1,\dots,n-1,\\ \partial^2_{z_j\bar z_k}\rho &\equiv 0 \quad\text{for } j\neq k,\ 1\leqslant j,k\leqslant n,\\ \partial^2_{z_n\bar z_n}\rho(z',z_n) &= \frac{1}{2}-\frac{1}{2}\left(\psi''(\Re z_n)\psi(\Re z_n)+\psi'(\Re z_n)^2\right),\end{aligned}$$

wherever ρ is of class $\mathcal{C}^2$ — which is the case for each $z:(z',z_n)\in\mathbb{C}^{n-1}\times\{z_n\mid 0<\Re(z_n)<A\}$. From this, the expression for $\mathscr{L}(\rho)(z;v)$ follows. ☐

4. General properties of visibility domains

In this section, we shall demonstrate three properties of visibility domains with respect to the Kobayashi distance. The first two results will seem to be of a slightly technical nature. However, the three results together form the crux of the argument underlying the observation — made in Section 1 — that several results that were shown for Goldilocks domains in [9] actually hold true for visibility domains with respect to the Kobayashi distance.

The proof of the first of these results, Proposition 4.1, is based on an argument developed by Karlsson in [20]. Its near-resemblance to Theorems 1.8 and 1.9 is suggestive. Of course, the conclusion (4.2) is weaker than what constitutes a Wolff–Denjoy-type theorem, but the domains appearing in Proposition 4.1 are — in contrast to those in the above-mentioned theorems — **merely** visibility domains with respect to the Kobayashi distance. Proposition 4.1 is an indication that some of the results stated in [9] for Goldilocks domains might be true for visibility domains with respect to the Kobayashi distance.

Proposition 4.1. *Let $\Omega\subset\mathbb{C}^n$ be a visibility domain with respect to the Kobayashi distance. Let $F:\Omega\to\Omega$ be a holomorphic map. Let $(\nu_j)_{j\geqslant 1}$ and $(\mu_j)_{j\geqslant 1}$ be two sequences of positive integers with $\nu_j,\mu_j\to\infty$. Suppose*

$$\lim_{j\to\infty}\mathsf{k}_\Omega(F^{\nu_j}(o),o)=\infty \quad \textit{and} \quad \lim_{j\to\infty}\mathsf{k}_\Omega(F^{\mu_j}(o),o)=\infty \tag{4.1}$$

for some $o \in \Omega$. Then there exists a $\xi \in \partial\Omega$ such that

$$\lim_{j\to\infty} F^{\nu_j}(z) = \xi = \lim_{j\to\infty} F^{\mu_j}(z) \tag{4.2}$$

for every $z \in \Omega$.

Proof. By (4.1) and the fact that Ω is bounded, we can find a subsequence $(\nu_{j_\ell})_{\ell\geqslant 1}$ such that:

(a) $\mathsf{k}_\Omega(F^{\nu_{j_\ell}}(o), o) \geqslant \mathsf{k}_\Omega(F^k(o), o)$ for every $k \leqslant \nu_{j_\ell}$, $\ell = 1, 2, 3, \dots$;
(b) $F^{\nu_{j_\ell}}(o) \to \xi$ for some point $\xi \in \partial\Omega$ as $\ell \to \infty$.

We now establish the following:

Claim: Let $z \in \Omega$ and let $(m_j)_{j\geqslant 1}$ be a sequence of positive integers with $m_j \to \infty$ such that $\mathsf{k}_\Omega(F^{m_j}(z), z) \to \infty$ as $j \to \infty$. Suppose $(m_{j_\ell})_{\ell\geqslant 1}$ is a subsequence such that

$$F^{m_{j_\ell}}(z) \to \eta \quad \text{as } \ell \to \infty,$$

where η is some point in $\partial\Omega$. Then $\eta = \xi$.

Proof of claim. We shall assume $\xi \neq \eta$ and aim for a contradiction. For simplicity of notation, let us, for just this paragraph, relabel $(\nu_{j_\ell})_{\ell\geqslant 1}$ as $(\nu_j)_{j\geqslant 1}$ — but with the understanding that it represents the subsequence introduced at the beginning of the proof. Also, relabel $(m_{j_\ell})_{\ell\geqslant 1}$ as $(m_j)_{j\geqslant 1}$. Pick a sequence $i_j \to \infty$ such that $\nu_{i_j} > m_j$, $j = 1, 2, 3, \dots$ Now let $\sigma_j : [0, T_j] \to \Omega$ be a $(1, 1)$-almost-geodesic with $\sigma_j(0) = F^{\nu_{i_j}}(o)$ and $\sigma_j(T_j) = F^{m_j}(z)$, whose existence is guaranteed by Proposition 4.4 of [9]. Since Ω is a visibility domain with respect to the Kobayashi distance, and as $\xi \neq \eta$ (by assumption), there exists an $R > 0$ so that

$$\sup_{j\geqslant 1} \mathsf{k}_\Omega(o, \sigma_j) \leqslant R,$$

where we write $\mathsf{k}_\Omega(o, \sigma_j) := \inf\{\mathsf{k}_\Omega(o, \sigma_j(t)) \mid t \in [0, T_j]\}$. We pick some $t_j \in [0, T_j]$ such that $\mathsf{k}_\Omega(o, \sigma_j(t_j)) \leqslant R$, $j = 1, 2, 3, \dots$ Then, by Lemma 2.9 we have

$$\begin{aligned}\mathsf{k}_\Omega(F^{\nu_{i_j}}(o), F^{m_j}(z)) &\geqslant \mathsf{k}_\Omega(F^{\nu_{i_j}}(o), \sigma_j(t_j)) + \mathsf{k}_\Omega(\sigma_j(t_j), F^{m_j}(z)) - 3\\ &\geqslant \mathsf{k}_\Omega(F^{\nu_{i_j}}(o), o) + \mathsf{k}_\Omega(o, F^{m_j}(z)) - 3 - 2R.\end{aligned}$$

On the other hand

$$\mathsf{k}_\Omega(F^{\nu_{i_j}}(o), F^{m_j}(z)) \leqslant \mathsf{k}_\Omega(F^{\nu_{i_j}-m_j}(o), o) + \mathsf{k}_\Omega(o, z) \leqslant \mathsf{k}_\Omega(F^{\nu_{i_j}}(o), o) + \mathsf{k}_\Omega(o, z).$$

The first inequality is due to the triangle inequality and the fact that F is contractive with respect to the Kobayashi distance and the second is due to the property (a). We conclude that

$$\begin{aligned}\mathsf{k}_\Omega(z, F^{m_j}(z)) &\leqslant \mathsf{k}_\Omega(F^{m_j}(z), F^{m_j}(o)) + \mathsf{k}_\Omega(F^{m_j}(o), o) + \mathsf{k}_\Omega(o, z)\\ &\leqslant 2R + 3 + 4\mathsf{k}_\Omega(o, z),\end{aligned} \tag{4.3}$$

which produces a contradiction. (The reader will notice that a more efficient bound is possible above, but we opt for the 3-term upperbound in (4.3) because we will need, and refer to, the idea behind this bound later.) Hence the claim. □

Taking $z = o$ in the above claim, and letting $(m_j)_{j\geqslant 1}$ represent any subsequence $(\nu_{j_k})_{k\geqslant 1}$ of $(\nu_j)_{j\geqslant 1}$ (respectively, $(\mu_{j_k})_{k\geqslant 1}$ of $(\mu_j)_{j\geqslant 1}$) for which $\big(F^{\nu_{j_k}}(o)\big)_{k\geqslant 1}$ is convergent (respectively, $\big(F^{\mu_{j_k}}(o)\big)_{k\geqslant 1}$ is convergent), we conclude that

$$\lim_{j\to\infty} F^{\nu_j}(o) = \xi = \lim_{j\to\infty} F^{\mu_j}(o).$$

Now consider $z \neq o$. Arguing as in (4.3) and by the fact that F is contractive, we have

$$\mathsf{k}_\Omega(F^{\nu_j}(z), z) \geqslant \mathsf{k}_\Omega(F^{\nu_j}(o), o) - 2\mathsf{k}_\Omega(o, z) \ \text{ and}$$
$$\mathsf{k}_\Omega(F^{\mu_j}(z), z) \geqslant \mathsf{k}_\Omega(F^{\mu_j}(o), o) - 2\mathsf{k}_\Omega(o, z).$$

Therefore, if $(m_j)_{j\geqslant 1}$ represents any subsequence $(\nu_{j_k})_{k\geqslant 1}$ of $(\nu_j)_{j\geqslant 1}$ (respectively, $(\mu_{j_k})_{k\geqslant 1}$ of $(\mu_j)_{j\geqslant 1}$) for which $\big(F^{\nu_{j_k}}(z)\big)_{k\geqslant 1}$ is convergent (respectively, $\big(F^{\mu_{j_k}}(z)\big)_{k\geqslant 1}$ is convergent), then from the last two inequalities and from (4.1), we have

$$\mathsf{k}_\Omega(F^{m_j}(z), z) \to \infty \quad \text{as } j \to \infty.$$

We can therefore appeal again to our claim, whence, arguing as above, we have (4.2). □

It turns out that in many applications of visibility, such as the Wolff–Denjoy-type theorems in this paper (as well as other applications which we shall address in forthcoming work), knowing that $\lim_{r\to 0^+} M_\Omega(r) = 0$ is of crucial importance. This is guaranteed, by definition, whenever Ω is a Goldilocks domain. It is not clear whether this is true for visibility domains with respect to the Kobayashi distance in general. However, for many sub-families of visibility domains, it can be shown that $\lim_{r\to 0^+} M_\Omega(r) = 0$. The following theorem is a result of this type.

Theorem 4.2. *Let Ω be a visibility domain with respect to the Kobayashi distance that is taut. Then* $\lim_{r\to 0^+} M_\Omega(r) = 0$.

Proof. Assume that $M_\Omega(r) \not\to 0$ as $r \to 0$. Since, by definition, $M_\Omega(r)$ is monotone, this implies that there exists a constant $\epsilon_0 > 0$ such that $M_\Omega(r) \searrow \epsilon_0$ as $r \searrow 0$. Thus, there exist a sequence of positive numbers $r_1 > r_2 > r_3 > \dots$ such that $r_\nu \to 0$ and, for each $\nu \in \mathbb{Z}_+$, a point $z_\nu \in \Omega$ such that $0 < \delta_\Omega(z_\nu) \leqslant r_\nu$ and such that:

- $\epsilon_0 \leqslant M_\Omega(r_\nu) < \epsilon_0 + 1/\nu$;
- $\exists v_\nu \in T^{1,0}_{z_\nu}\Omega$ satisfying $\|v_\nu\| = 1$ and

$$\frac{1}{\kappa_\Omega(z_\nu; v_\nu)} > \epsilon_0 - \frac{1}{\nu}. \tag{4.4}$$

Owing to the definition of κ_Ω, (4.4) implies that there exists, for each $\nu \in \mathbb{Z}_+$, a holomorphic map $\varphi_\nu \in \mathcal{O}(\mathbb{D}; \Omega)$ satisfying

$$\varphi_\nu(0) = z_\nu, \quad \varphi_\nu'(0) \in \{t \cdot v_\nu \mid t > 0\} \quad \text{and} \quad \|\varphi_\nu'(0)\| \geqslant \epsilon_0 - 1/\nu.$$

Passing to a subsequence and relabelling if necessary, we may assume:

(a) There exists a point $\xi \in \partial\Omega$ such that $z_\nu \to \xi$;
(b) There exists a map $\varphi \in \mathcal{O}(\mathbb{D}; \overline{\Omega})$ such that $\varphi_\nu \to \varphi$ uniformly on compact subsets.

The conclusion (b) is a consequence of Montel's theorem. However, as $z_\nu \to \xi \in \partial\Omega$, it follows from the tautness of Ω that $\varphi(\mathbb{D}) \subset \partial\Omega$.

It follows from the above discussion that $\|\varphi'(0)\| \geqslant \epsilon_0$. However, as $\varphi'_\nu \to \varphi'$ uniformly on compact sets also, and as — owing to the fact that Ω is taut — $\kappa_\Omega : \Omega \times \mathbb{C}^n \to [0, +\infty)$ is continuous, see Result 2.2-(1), there exist a small constant $\delta_1 > 0$ and a number $N_1 \in \mathbb{Z}_+$ such that

$$\|\varphi'(\zeta)\|, \ \|\varphi'_\nu(\zeta)\| \geqslant \epsilon_0/2 \quad \forall \zeta \in \overline{D(0, \delta_1)} \text{ and } \forall \nu \geqslant N_1, \tag{4.5}$$

$$\kappa_\Omega\big(\varphi_\nu(\zeta); \varphi'_\nu(\zeta)\big) \leqslant 2/\epsilon_0 \quad \forall \zeta \in \overline{D(0, \delta_1)} \text{ and } \forall \nu \geqslant N_1. \tag{4.6}$$

Let π_j denote the projection onto the j-th coordinate. By Cauchy's estimates, the magnitude of each of the derivatives of $\pi_j \circ \varphi_\nu$, $j = 1, \dots, n$, $\nu \geqslant N_1$, is bounded above by a quantity that depends only on $\sup_{x:|x|=\delta_1} |\pi_j \circ \varphi_\nu(x)|$, δ_1, and the order of the derivative in question, and which is independent of ζ if $\zeta \in \overline{D(0; \delta_1/2)}$. Thus, by a standard power-series argument and by (4.5), we can find a small constant $\delta_2 \in (0, \delta_1/2)$ and an integer $N_2 \geqslant N_1$ so that

$$\|\varphi_\nu(\zeta_1) - \varphi_\nu(\zeta_2)\| \geqslant \frac{\epsilon_0}{4}|\zeta_1 - \zeta_2| \quad \forall \zeta_1, \zeta_2 \in \overline{D(0, \delta_2)} \text{ and } \forall \nu \geqslant N_2. \tag{4.7}$$

Let us now write

$$\partial\Omega \ni \eta := \varphi(\delta_2/2) \quad \text{and} \quad w_\nu := \varphi_\nu(\delta_2/2).$$

It follows from (4.7) that $\xi \neq \eta$. Clearly, $w_\nu \to \eta$. Let $\gamma : ([0, T], 0, T) \to (\mathbb{D}, 0, \delta_2/2)$ denote the geodesic with respect to the Poincaré distance on $\mathbb{D}$ from 0 to $\delta_2/2$ that lies in $[0, 1) \subset \mathbb{D}$. Let us define $\sigma_\nu : [0, T] \to \Omega$ as $\sigma_\nu(t) := \varphi_\nu \circ \gamma(t)$. We claim that each σ_ν, $\nu \geqslant N_2$, is a $(\lambda, 0)$-almost geodesic for an appropriate $\lambda \geqslant 1$. We first note that as γ is the restriction of a diffeomorphic embedding of $\mathbb{R}$ into $\mathbb{D}$, there exists a constant $r_0 > 0$ such that

$$|\gamma(s) - \gamma(t)| \geqslant r_0|s - t| \quad \forall s, t \in [0, T]. \tag{4.8}$$

We now estimate, for any $s, t \in [0, T]$ and any $\nu \geqslant N_2$:

$$\begin{aligned} \mathsf{k}_\Omega\big(\sigma_\nu(s), \sigma_\nu(t)\big) &\geqslant c\|\sigma_\nu(s) - \sigma_\nu(t)\| \\ &\geqslant \frac{c\,\epsilon_0}{4}|\gamma(s) - \gamma(t)| && \text{(by (4.7) above)} \\ &\geqslant \frac{c\,\epsilon_0\, r_0}{4}|s - t| && \text{(by (4.8) above)} \end{aligned}$$

Here, the constant $c > 0$ in the first inequality is as given by Result 2.1. On the other hand, by the fact that each φ_ν is contractive relative to the Kobayashi distance, we have for any $s, t \in [0, T]$ (recall that the Poincaré distance on $\mathbb{D}$ is $\mathsf{k}_{\mathbb{D}}$):

$$\begin{aligned} \mathsf{k}_\Omega\big(\sigma_\nu(s), \sigma_\nu(t)\big) &\leqslant \mathsf{k}_{\mathbb{D}}\big(\gamma(s), \gamma(t)\big) \\ &= |s-t|. \end{aligned}$$

Furthermore, by (4.6), we have

$$\kappa_\Omega\big(\sigma_\nu(t); \sigma_\nu'(t)\big) \leqslant \frac{2}{\epsilon_0} \sup\nolimits_{\tau\in[0,T]} |\gamma'(\tau)|.$$

From these estimates, and by the fact that as each σ_ν — being $\mathcal{C}^\infty$-smooth — is absolutely continuous, we get that each σ_ν, $\nu \geqslant N_2$, is a $(\lambda, 0)$-almost geodesic from z_ν to w_ν with

$$\lambda = \max\left(1, \frac{4}{c\,\epsilon_0\, r_0}, \frac{2}{\epsilon_0} \sup\nolimits_{\tau\in[0,T]} |\gamma'(\tau)|\right).$$

Since $\varphi(\mathbb{D}) \subset \partial\Omega$, it follows that given any compact subset $K \subset \Omega$ there exists an integer $N_K \gg 1$ such that $\varphi_\nu(\,\overline{D(0, \delta_2/2)}\,) \cap K = \varnothing$ for every $\nu \geqslant N_K$. But this, together with our conclusions about σ_ν (for $\nu \geqslant N_2$), contradicts the fact that Ω is a visibility domain with respect to the Kobayashi distance. Hence our assumption about M_Ω must be false. □

The last result in this section is one whose conclusion identifies a property that is possessed by domains with smooth boundaries that are "sufficiently curved" in a certain sense. However, Theorem 4.3 establishes that any taut visibility domain — whose boundary is, in general far less well-behaved — also has the desirable property alluded to.

Theorem 4.3. *Let X be a connected complex manifold and let $\Omega \subset \mathbb{C}^n$ be a visibility domain with respect to the Kobayashi distance that is taut. Suppose $(\varphi_\nu)_{\nu\geqslant 1}$ is a sequence in $\mathcal{O}(X;\Omega)$ that converges uniformly on compacts of X to a holomorphic map $\psi : X \to \partial\Omega$. Then ψ is a constant map.*

Proof. Fix $x \in X$. For any $f \in \mathcal{O}(X;\Omega)$, let f' denote the holomorphic total derivative of f. Since $(\varphi_\nu)_{\nu\geqslant 1}$ converges uniformly on compacts to ψ, it follows that $\varphi_\nu'(x) \to \psi'(x)$ (the easiest way to understand this is to equip X with some hermitian metric; the choice of metric is irrelevant to the proof). Fix a vector $v_0 \in (T_x^{1,0}X)\setminus\{0\}$. We claim that, given a $\nu \in \mathbb{Z}_+$, $\|\varphi_\nu'(x)v_0\| \leqslant \kappa_X(x, v_0) M_\Omega\big(\delta_\Omega(\varphi_\nu(x))\big)$. There is nothing to prove if $v_0 \in \operatorname{Ker}(\varphi_\nu'(x))$. Thus, assume that $v_0 \notin \operatorname{Ker}(\varphi_\nu'(x))$. We estimate:

$$\frac{\|\varphi_\nu'(x)v_0\|}{\kappa_\Omega\big(\varphi_\nu(x); \varphi_\nu'(x)v_0\big)} = \frac{1}{\kappa_\Omega\left(\varphi_\nu(x); \frac{\varphi_\nu'(x)v_0}{\|\varphi_\nu'(x)v_0\|}\right)} \leqslant M_\Omega\big(\delta_\Omega(\varphi_\nu(x))\big).$$

The inequality on the right side is due to the definition of M_Ω. Therefore

$$\|\varphi_\nu'(x)v_0\| \leqslant \kappa_\Omega\big(\varphi_\nu(x); \varphi_\nu'(x)v_0\big)M_\Omega\big(\delta_\Omega(\varphi_\nu(x))\big) \leqslant \kappa_X(x; v_0)M_\Omega\big(\delta_\Omega(\varphi_\nu(x))\big),$$

which is the desired claim. The second inequality is due the metric-decreasing property of holomorphic maps. By hypothesis, $\delta_\Omega(\varphi_\nu(x)) \to 0$ as $\nu \to \infty$. Since Ω is taut, it follows from Theorem 4.2 that $M_\Omega\big(\delta_\Omega(\varphi_\nu(x))\big) \to 0$ as $\nu \to \infty$. Therefore, from the last inequality, we see that $\varphi_\nu'(x)v_0 \to 0$ as $\nu \to \infty$. This in turn implies that $\psi'(x)v_0 = 0$. Now $v_0 \in (T_x^{1,0}X) \setminus \{0\}$ was arbitrary, whence we get $\psi'(x) \equiv 0$. As the above $x \in X$ was arbitrary, and as X is connected, it follows that ψ is a constant. □

5. The proof of Theorem 1.5

This section is devoted to proving Theorem 1.5. To do so, we first need a technical lemma.

Lemma 5.1. *Let f be as in Theorem* 1.5. *Fix constants $\lambda \geqslant 1$ and $\kappa \geqslant 0$. Then, given $\epsilon > 0$, there exist constants $-\infty < a' < b' < +\infty$ such that*

$$\int_{-\infty}^{a'} M_\Omega\Big(\frac{1}{f^{-1}\big((1/2\lambda)|t| - (\kappa/2)\big)}\Big)dt < \epsilon,$$
$$\int_{b'}^{+\infty} M_\Omega\Big(\frac{1}{f^{-1}\big((1/2\lambda)t - (\kappa/2)\big)}\Big)dt < \epsilon.$$

Proof. The result is a consequence of the change-of-variable formula, using

$$r := \frac{1}{f^{-1}\big((1/2\lambda)|t| - (\kappa/2)\big)}$$

for the first integral, and

$$r := \frac{1}{f^{-1}\big((1/2\lambda)t - (\kappa/2)\big)}$$

for the second. We omit the routine computations that these changes of variable necessitate. The inequalities follow from the integrability condition (1.2). □

We are now in a position to give the:

Proof of Theorem 1.5. We proceed by contradiction. Assume thus that there exist constants $\lambda \geqslant 1$ and $\kappa \geqslant 0$, a pair of distinct points $\xi, \eta \in \partial\Omega$, neighbourhoods V and W of ξ and η, respectively, in $\overline{\Omega}$ with $\overline{V} \cap \overline{W} = \varnothing$, and a sequence $(\sigma_\nu)_{\nu\geqslant 1}$ of

(λ, κ)-almost-geodesics, $\sigma_\nu : [a_\nu, b_\nu] \to \Omega$, such that $\sigma_\nu(a_\nu) \in V$ and $\sigma_\nu(b_\nu) \in W$ for all ν and such that

$$\max_{t\in[a_\nu,b_\nu]} \delta_\Omega(\sigma_\nu(t)) \to 0 \text{ as } \nu \to \infty.$$

By re-parametrizing, we can assume that, for all ν, $a_\nu \leqslant 0 \leqslant b_\nu$ and that

$$\delta_\Omega(\sigma_\nu(0)) = \max_{t\in[a_\nu,b_\nu]} \delta_\Omega(\sigma_\nu(t)).$$

By Result 2.8, there exists a $C < \infty$ such that every Ω-valued (λ, κ)-almost-geodesic is C-Lipschitz with respect to the Euclidean distance. Therefore, by using the Arzela–Ascoli theorem and passing to an appropriate subsequence, we may assume:

- $a_\nu \to a \in [-\infty, 0]$ and $b_\nu \to b \in [0, +\infty]$;
- $(\sigma_\nu)_{\nu\geqslant 1}$ converges locally uniformly on (a, b) to a continuous map $\sigma : (a, b) \to \overline{\Omega}$;
- $(\sigma_\nu(a_\nu))_{\nu\geqslant 1}$ converges to $\xi' \in \overline{V}$;
- $(\sigma_\nu(b_\nu))_{\nu\geqslant 1}$ converges to $\eta' \in \overline{W}$.

Clearly, $\xi' \neq \eta'$ because $\overline{V} \cap \overline{W} = \varnothing$. We can conclude from the fact that

$$\|\sigma_\nu(a_\nu) - \sigma_\nu(b_\nu)\| \leqslant C(b_\nu - a_\nu) \quad \forall \nu \in \mathbb{Z}_+$$

that $a < b$.

Claim: If $\theta : [s_1, s_2] \to \Omega$ is a (λ, κ)-almost-geodesic, then for almost every $t \in [s_1, s_2]$, $\|\theta'(t)\| \leqslant \lambda M_\Omega(\delta_\Omega(\theta(t)))$.

Proof of claim. By the definition of a (λ, κ)-almost-geodesic we have $\kappa_\Omega(\theta(t), \theta'(t)) \leqslant \lambda$ for almost every $t \in [s_1, s_2]$. If $\theta'(t) = 0$, then the claim is trivially true. If $\theta'(t) \neq 0$, we have

$$\kappa_\Omega\left(\theta(t), \frac{\theta'(t)}{\|\theta'(t)\|}\right) \leqslant \frac{\lambda}{\|\theta'(t)\|}.$$

So

$$\|\theta'(t)\| \leqslant \lambda \cdot \frac{1}{\kappa_\Omega\Big(\theta(t), \frac{\theta'(t)}{\|\theta'(t)\|}\Big)} \leqslant \lambda M_\Omega\big(\delta_\Omega(\theta(t))\big).$$

We first assert that $\sigma : (a, b) \to \overline{\Omega}$ is constant. To prove this, we use the fact that $M_\Omega(t) \searrow 0$ as $t \searrow 0$. With this, the proof proceeds exactly along the lines of the proof of Claim 1 in [9, Section 5]. Hence, we omit the proof.

We shall now show that σ is *not* constant. Our argument involves the study of two cases.

Case 1. Both a and b are finite.

In this case, we first define the C-Lipschitz maps $\widetilde{\sigma}_\nu : [a, b] \to \Omega$ obtained by restricting each σ_ν to $[a_\nu, b_\nu] \cap [a, b]$ and then extending the restricted map continuously to $[a, b]$ by defining the extension to be a constant on the intervals $[a, a_\nu]$ and $[b_\nu, b]$ whenever $a < a_\nu$ or $b_\nu < b$. We can then infer by a standard argument that σ extends to a continous map $\widetilde{\sigma} : [a, b] \to \overline{\Omega}$. We have $\widetilde{\sigma}(a) = \xi' \neq \eta' = \widetilde{\sigma}(b)$. By continuity of $\widetilde{\sigma}$, it follows that $\widetilde{\sigma}|_{(a,b)}$ is non-constant.

Case 2. Either $a = -\infty$ or $b = +\infty$.

We make a couple of preliminary observations. For every $\nu \in \mathbb{Z}_+$ and every $t \in [a_\nu, b_\nu]$,

$$\frac{1}{\lambda}|t| - \kappa \leqslant \mathsf{k}_\Omega(\sigma_\nu(0), \sigma_\nu(t)) \leqslant \mathsf{k}_\Omega(\sigma_\nu(0), z_0) + \mathsf{k}_\Omega(z_0, \sigma_\nu(t)) \leqslant 2f\left(\frac{1}{\delta_\Omega(\sigma_\nu(t))}\right),$$

because $\delta_\Omega(\sigma_\nu(0)) \geqslant \delta_\Omega(\sigma_\nu(t))$.

Let us first consider the case when $b = +\infty$. By the properties of the sequence $(\sigma_\nu)_{\nu\geqslant 1}$, it follows that there exists $N \in \mathbb{Z}_+$ and a constant $B \gg 1$ such that

$$\frac{1}{2\lambda}|t| - \frac{\kappa}{2} \in \mathsf{range}(f) \quad \forall t \in (B, b_\nu] \text{ and } \forall \nu \geqslant N.$$

Thus, by the fact that f is strictly increasing, we get:

$$f^{-1}\left(\frac{1}{2\lambda}|t| - \frac{\kappa}{2}\right) \leqslant \frac{1}{\delta_\Omega(\sigma_\nu(t))} \quad \forall t \in (B, b_\nu] \text{ and } \forall \nu \geqslant N, \tag{5.1}$$

in case $b = +\infty$. If $a = -\infty$, we can argue in exactly the same way to find a constant $A \gg 1$ such that

$$f^{-1}\left(\frac{1}{2\lambda}|t| - \frac{\kappa}{2}\right) \leqslant \frac{1}{\delta_\Omega(\sigma_\nu(t))} \quad \forall t \in [a_\nu, -A) \text{ and } \forall \nu \geqslant N \tag{5.2}$$

(where N is exactly as above).

At this juncture, we shall assume that $a = -\infty$ and $b = +\infty$. This is the principal sub-case; we shall merely indicate the changes that would be needed in the argument that follows in case either **one** of a or b is finite. With this assumption, we have, by monotonicity of M_Ω and from (5.1) and (5.2):

$$M_\Omega\big(\delta_\Omega(\sigma_\nu(t))\big) \leqslant M_\Omega\left(\frac{1}{f^{-1}\big((1/2\lambda)|t| - (\kappa/2)\big)}\right)$$

for evey $t \in [a_\nu, -A) \cup (B, b_\nu]$ and for every $\nu \geqslant N$. So, finally, by our claim above, we conclude that

$$\begin{aligned}\|\sigma_\nu'(t)\| &\leqslant \lambda M_\Omega\big(\delta_\Omega(\sigma_\nu(t))\big)\\ &\leqslant \lambda M_\Omega\Big(\frac{1}{f^{-1}\big((1/2\lambda)|t| - (\kappa/2)\big)}\Big) \\ &\quad \text{for a.e. } t \in [a_\nu, -A) \cup (B, b_\nu] \text{ and } \forall \nu \geqslant N.\end{aligned} \tag{5.3}$$

Using Lemma 5.1, we choose $a' \in (-\infty, -A)$ and $b' \in (B, +\infty)$ such that

$$\lambda \int_{-\infty}^{a'} M_\Omega\Big(\frac{1}{f^{-1}\big((1/2\lambda)|t| - (\kappa/2)\big)}\Big) dt + \lambda \int_{b'}^{+\infty} M_\Omega\Big(\frac{1}{f^{-1}\big((1/2\lambda)t - (\kappa/2)\big)}\Big) dt < \|\xi' - \eta'\|.$$

Then

$$\begin{aligned}
&\|\sigma(b') - \sigma(a')\| \\
&= \lim_{\nu\to\infty} \|\sigma_\nu(b') - \sigma_\nu(a')\| \\
&\geqslant \limsup_{\nu\to\infty} \big(\|\sigma_\nu(b_\nu) - \sigma_\nu(a_\nu)\| - \|\sigma_\nu(a_\nu) - \sigma_\nu(a')\| - \|\sigma_\nu(b_\nu) - \sigma_\nu(b')\|\big) \\
&\geqslant \|\xi' - \eta'\| - \limsup_{\nu\to\infty} \int_{a_\nu}^{a'} \|\sigma_\nu'(t)\| dt - \limsup_{\nu\to\infty} \int_{b'}^{b_\nu} \|\sigma_\nu'(t)\| dt \\
&\geqslant \|\xi' - \eta'\| - \limsup_{\nu\to\infty} \lambda \int_{a_\nu}^{a'} M_\Omega\Big(\frac{1}{f^{-1}\big((1/2\lambda)|t| - (\kappa/2)\big)}\Big) dt \\
&\quad - \limsup_{\nu\to\infty} \lambda \int_{b'}^{b_\nu} M_\Omega\Big(\frac{1}{f^{-1}\big((1/2\lambda)t - (\kappa/2)\big)}\Big) dt \qquad \text{(using (5.3))} \\
&= \|\xi' - \eta'\| - \lambda \int_{-\infty}^{a'} M_\Omega\Big(\frac{1}{f^{-1}\big((1/2\lambda)|t| - (\kappa/2)\big)}\Big) dt \\
&\quad - \lambda \int_{b'}^{+\infty} M_\Omega\Big(\frac{1}{f^{-1}\big((1/2\lambda)t - (\kappa/2)\big)}\Big) dt \\
&> 0.
\end{aligned}$$

This shows that σ is not constant.

If a is finite, then by an analogue of the argument described in Case 1 (by constructing auxiliary maps that are C-Lipschitz on $[a, 0]$), we infer that σ extends to a continuous map $\widetilde{\sigma} : [a, +\infty)$. We now estimate $\|\sigma(b') - \widetilde{\sigma}(a)\|$ — with $b' \in (B, +\infty)$ chosen appropriately so that we can argue as in the previous paragraph — to get $\|\sigma(b') - \widetilde{\sigma}(a)\| > 0$. An analogous description can be given for the argument in case b is finite. This completes the argument for Case 2, with the conclusion that σ is not constant.

This last assertion produces a contradiction. Thus, the assumption made at the beginning must be false, which completes the proof. □

6. A family of planar comparison domains

In this section we take the first step in showing that caltrops have the properties stated in Theorem 1.5. The essential idea is as follows: we first explicitly calculate the Kobayashi distance on a model planar domain D. Then, given a caltrop $\Omega \subset \mathbb{C}^n$, $n \geq 2$, we shall affinely embed copies of D into Ω in such a way that every point of Ω that is sufficiently close to $\partial\Omega$ is contained in one of these embedded domains. Then, the distance-decreasing property of holomorphic mappings for the Kobayashi distance could be used to estimate the Kobayashi distance on Ω.

Given the geometry of the boundary of a caltrop, the model comparison domain D that we need will be bounded, symmetric about the real axis, have 0 as a boundary point and the tip of an outward-pointing cusp. In fact, it will be useful to construct a family of model planar domains having the latter properties. To this end, given $a, h > 0$, define the following domains in $\mathbb{C}$:

$$S_{a,h} := \{z = x + iy \in \mathbb{C} \mid x > a,\ -h < y < h\}. \tag{6.1}$$

Let us denote the domains that we are interested in by $\mathcal{Q}^{\alpha,a,h}$, where $\mathcal{Q}^{\alpha,a,h}$ is the image of $S_{a,h}$ under the following biholomorphisms, composed in the order given below:

$$\begin{aligned} \mathsf{inv}(z) &:= \frac{1}{z} \quad \forall z \in \mathbb{C} \setminus \{0\}, \\ \phi_\alpha(z) &:= z^\alpha \quad \forall z \in \mathsf{inv}(S_{a,h}). \end{aligned}$$

Here α is a real number greater than 1, and $a, h > 0$ are such that ϕ_α is in fact a biholomorphism. That $a, h > 0$ can be so chosen follows from an elementary calculation. Specifically, we compute:

$$\mathsf{inv}(S_{a,h}) = \left(\mathbb{C} \setminus \overline{D(-i/2h, 1/2h)}\right) \cap \left(\mathbb{C} \setminus \overline{D(i/2h, 1/2h)}\right) \cap D(1/2a, 1/2a).$$

Let us denote $\mathsf{inv}(S_{a,h})$ by $T_{a,h}$.

We make a simple observation which will be useful in the proposition below. The region $T_{a,h}$ contains 0 in its boundary and has a quadratic cusp at 0. This means that there exist constants $c_1, c_2 > 0$ such that, for every $z \in \partial T_{a,h}$,

$$\begin{aligned} c_1\Re(z)^2 &\leqslant \Im(z) \leqslant c_2\Re(z)^2, \text{ or} \\ -c_2\Re(z)^2 &\leqslant \Im(z) \leqslant -c_1\Re(z)^2, \end{aligned} \tag{6.2}$$

depending on whether $\Im(z) \geqslant 0$ or $\Im(z) \leqslant 0$, provided $\Re(z)$ is sufficiently small. In fact, by straightforward calculations we see that for some $\delta > 0$ sufficiently small, $\partial T_{a,h} \cap \{z \in \mathbb{C} \mid 0 \leqslant \Re(z) \leqslant \delta\} = \mathsf{gr}(f) \cup \mathsf{gr}(-f)$, where

$$f(x) = hx^2 + O(x^4) \ \text{ as } x \to 0^+, \tag{6.3}$$

with the understanding that $z = x + iy$. In this section, $\mathsf{gr}(\cdot)$ will denote the graph of a specified function.

The following proposition describes the features of the (family of) domains $Q^{\alpha,a,h}$ that will be relevant to estimating Kobayashi distances — in the manner hinted at above — on caltrops.

Proposition 6.1. *Fix $\alpha > 1$ and let $T_{a,h}$ and $Q^{\alpha,a,h}$ be as described above — with $a, h > 0$ appropriately chosen. Set $p := (1+\alpha)/\alpha$. Then:*

(1) *There exist constants $\epsilon, C_1, C_2 > 0$ such that, for every $z \in \partial Q^{\alpha,a,h} \cap \{z \in \mathbb{C} \mid 0 \leqslant \Re(z) \leqslant \epsilon\}$, we have*

$$C_1\Re(z)^p \leqslant \Im(z) \leqslant C_2\Re(z)^p, \ \textit{or}$$
$$-C_2\Re(z)^p \leqslant \Im(z) \leqslant -C_1\Re(z)^p,$$

depending on whether $\Im(z) \geqslant 0$ or $\Im(z) \leqslant 0$.

(2) *If we fix a constant $M \geqslant 2$, then we can choose an $\epsilon > 0$ sufficiently small so that the inequalities in* (6.1) *hold true with $C_2 := Mh\alpha$. Furthermore, for a given $\alpha > 1$ and $h > 0$, this choice of ϵ decreases as $a \nearrow +\infty$.*

(3) *Fix some point $x_0 \in Q^{\alpha,a,h} \cap \mathbb{R}$. There exists a constant $C > 0$, which depends on x_0, such that*

$$\mathsf{k}_{Q^{\alpha,a,h}}(x_0, x) \leqslant C + \frac{\pi}{4h}x^{-1/\alpha} \quad \forall x \in (0, x_0).$$

Proof. To prove (6.1), we must examine the image of $T_{a,h}$ under ϕ_α close to $0 \in \partial T_{a,h}$. Let c_1, c_2 be the constants given by (6.2), and let the function f be as introduced just prior to (6.3). We examine the images of $\mathsf{gr}(f)$ and $\mathsf{gr}(-f)$ under ϕ_α. Let us, for example, examine the image of $\mathsf{gr}(f)$ under ϕ_α. An arbitrary element of $\mathsf{gr}(f)$ that is close to 0 can be written as $x + iy$, where $x \geqslant 0$ and $c_1x^2 \leqslant y \leqslant c_2x^2$. For $x > 0$ and sufficiently small, we compute:

$$\begin{aligned}\phi_\alpha(z) &= (x+iy)^\alpha \\ &= x^\alpha\left(1 + \sum_{j=1}^{\infty}\frac{(-1)^j}{(2j)!}\prod_{\nu=0}^{2j-1}(\alpha-\nu)\frac{y^{2j}}{x^{2j}}\right) \\ &\quad + ix^\alpha\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\prod_{\nu=0}^{2j}(\alpha-\nu)\frac{y^{2j+1}}{x^{2j+1}}\right).\end{aligned}$$

Using the fact that $c_1x^2 \leqslant y \leqslant c_2x^2$, it is easy to see that

$$\Re(\phi_\alpha(z)) = x^\alpha + O(x^{2+\alpha}),$$
$$c_1\alpha x^{1+\alpha}(1 - O(x^2)) \leqslant \Im(\phi_\alpha(z)) \leqslant c_2\alpha x^{1+\alpha}(1 + O(x^2))$$

for $z = x + iy \in \mathsf{gr}(f)$ and for $x > 0$ sufficiently small.

It follows from this that we can find constants $\epsilon, C_1, C_2 > 0$ such that for all $w \in \partial\mathcal{Q}^{\alpha,a,h}$ with $\Re(w) \leqslant \epsilon$ and $\Im(w) \geqslant 0$,

$$C_1\big(\Re(w)\big)^p \leqslant \Im(w) \leqslant C_2\big(\Re(w)\big)^p.$$

From this and the fact that, if $z \in \partial T_{a,h} \cap \{\Im(z) \leqslant 0\}$, then $z \in \mathsf{gr}(-f)$ (when $\Re(z)$ is sufficiently small), part (1) follows.

Part (2) is elementary and follows from the manner in which $\mathsf{dom}(f)$, by construction, depends on a, from (6.3), and from the estimates in the last paragraph.

We now address the Kobayashi-distance inequality that we need. We have a biholomorphism $\Phi_{\alpha,a,h}$ from $\mathcal{Q}^{\alpha,a,h}$ onto $\mathbb{D}$, given by

$$\Phi_{\alpha,a,h} = f_4 \circ f_3 \circ f_2 \circ f_1 \circ \mathsf{inv} \circ \big(\phi_\alpha|_{T_{a,h}}\big)^{-1},$$

where

$$f_1(z) = i(z - a) \quad \forall z \in S_{a,h},$$
$$f_2(z) = \frac{\pi z}{2h} \quad \forall z \in \{w \in \mathbb{C} \mid -h < \Re(w) < h,\ \Im(w) > 0\},$$
$$f_3(z) = \sin(z) \quad \forall z \in \{w \in \mathbb{C} \mid -\pi/2 < \Re(w) < \pi/2,\ \Im(w) > 0\},$$
$$f_4(z) = \frac{z - i}{z + i} \quad \forall z \in \{w \in \mathbb{C} \mid \Im(w) > 0\}.$$

The explicit expression for $\Phi_{\alpha,a,h}$ is

$$\Phi_{\alpha,a,h}(z) = \frac{\sin\Big(\frac{\pi i}{2h}\big(\frac{1}{z^{1/\alpha}} - a\big)\Big) - i}{\sin\Big(\frac{\pi i}{2h}\big(\frac{1}{z^{1/\alpha}} - a\big)\Big) + i} \quad \forall z \in \mathcal{Q}^{\alpha,a,h}. \tag{6.4}$$

Observe that $\Phi_{\alpha,a,h}$ maps the closed and bounded interval $\overline{\mathcal{Q}^{\alpha,a,h} \cap \mathbb{R}}$ homeomorphically onto $[-1, 1]$. Furthermore, it is easy to check that $\Phi_{\alpha,a,h}$ maps the point

$$o_{\alpha,a,h} := o := 1/\big((2h/\pi)\log(\sqrt{2} + 1) + a\big)^\alpha \tag{6.5}$$

of $\mathcal{Q}^{\alpha,a,h}$ to 0 and that if $x \in \mathcal{Q}^{\alpha,a,h} \cap \mathbb{R}$ is less than o then $\Phi_{\alpha,a,h}(x) \in (0, 1)$. Therefore, for all such x,

$$\mathsf{k}_{\mathcal{Q}^{\alpha,a,h}}(o, x) = \mathsf{k}_{\mathbb{D}}\big(0, \Phi_{\alpha,a,h}(x)\big) = \frac{1}{2}\log\left(\frac{1 + \Phi_{\alpha,a,h}(x)}{1 - \Phi_{\alpha,a,h}(x)}\right).$$

Using (6.4), we obtain

$$\begin{aligned}
&\frac{1}{2}\log\left(\frac{1+\Phi_{\alpha,a,h}(x)}{1-\Phi_{\alpha,a,h}(x)}\right)\\
&=\frac{1}{2}\log\left(\exp\left\{\frac{\pi}{2h}\Big(\frac{1}{x^{1/\alpha}}-a\Big)\right\}-\exp\left\{-\frac{\pi}{2h}\Big(\frac{1}{x^{1/\alpha}}-a\Big)\right\}\right)-\frac{\log 2}{2}\\
&\leqslant\frac{1}{2}\log\left(\exp\left\{\frac{\pi}{2h}\Big(\frac{1}{x^{1/\alpha}}-a\Big)\right\}\right)\\
&\leqslant\frac{\pi}{4h}x^{-1/\alpha}.
\end{aligned}$$

From this and the triangle inequality, (6.1) of our proposition follows. □

The next few lemmas establish some basic observations that will — given a caltrop $\Omega\subset\mathbb{C}^n$, $n\geq 2$ — enable us to affinely embed copies of $\mathcal{Q}^{\alpha,a,h}$, for suitable choices of the parameters α, a and h, into Ω in the manner hinted at in the beginning of this section. (The actual estimates showing that caltrops possess the properties stated in the General Visibility Lemma will be obtained in the next section.) A note about our notation: in the lemmas that follow, the point o will be as introduced in (6.5), and will be associated to the specific $\mathcal{Q}^{\alpha,a,h}$ occurring in each lemma. Also, the lemmas below hold true for the parameter $p\in(1,2)$, and will be proved as such. In the next section, where caltrops make an appearance, we shall restrict p to $(1,3/2)$.

Lemma 6.2. *Suppose $\epsilon>0$ and $\phi:[0,\epsilon)\to\mathbb{R}$ is a continuous, strictly increasing function that is differentiable on $(0,\epsilon)$, such that ϕ' is increasing and such that $\phi(0)=0$. Then, for every $(x,y)\in[0,+\infty)\times[0,+\infty)$ such that $x+y<\epsilon$, $\phi(x+y)\geqslant\phi(x)+\phi(y)$.*

The proof of the above lemma is an elementary exercise in calculus.

Lemma 6.3. *Let $\psi:[0,A]\to[0,+\infty)$ be a continuous function that is $\mathcal{C}^2$ on $(0,A)$, where $A>0$. Let $p\in(1,2)$. Assume furthermore that:*

- *There exists a constant $C>1$ such that*

$$(1/C)x^p\leqslant\psi(x)\leqslant Cx^p\quad\forall x\in[0,A];$$

- *ψ is strictly increasing;*
- *ψ' is increasing on $(0,A)$.*

Write $\mathcal{R}:=\{z\in\mathbb{C}\mid 0<\Re(z)<A,\ |\Im(z)|<\psi(\Re(z))\}$. Then there exist a constant $B\in(0,A)$, a compact subset K that intersects $\{z\in\mathbb{C}\mid\Re(z)=A\}$ and such that $K\setminus\{z\in\mathbb{C}\mid\Re(z)=A\}\subsetneq\mathcal{R}$, and constants $a,h>0$ such that for each $x+iy\in\mathcal{R}$ with $x\leqslant B$, we have

(1) $(\psi^{-1}(|y|)+iy)+\mathcal{Q}^{1/(p-1),a,h}\subseteq\mathcal{R}$;
(2) $\psi^{-1}(|y|)+o>x$;

(3) $(\psi^{-1}(|y|)+iy)+o \in K$;
(4) $\delta_{\mathcal{R}}(x+iy) \leqslant |\psi^{-1}(|y|)-x|$.

Proof. It follows from Proposition 6.1 and the observation made prior to it that we may fix a constant $M \geqslant 2$ such that for every $\alpha > 1$ and every $a, h > 0$ there exists an $\epsilon \equiv \epsilon(\alpha, a, h) > 0$ such that

$$\mathcal{Q}^{\alpha,a,h} \subset \{w \in \mathbb{C} \mid 0 < \Re(w) < \epsilon,\ |\Im(w)| < Mh\alpha(\Re(w))^{(1+\alpha)/\alpha}\} =: S^{\alpha,a,h}$$

and such that, for any given $\alpha > 1$ and $h > 0$, $\epsilon \to 0$ as $a \to +\infty$. We let $\alpha := 1/(p-1)$. We note that, by the geometry of $\mathcal{Q}^{\alpha,a,h}$, the ϵ with the above properties does not decrease as we decrease h. Hence, we can choose a and h such that $\epsilon < A/2$ and $Mh\alpha < 1/C$. Now, fix a constant B, $0 < B < A$ so that

$$B < \min(o, (\epsilon/2)).$$

Let $z = x+iy \in \mathcal{R}$ and $x \leqslant B$. We consider the set $\mathcal{Q}^{\alpha,a,h} + (\psi^{-1}(|y|)+iy)$. An arbitrary element of this set is of the form $(\psi^{-1}(|y|)+s)+i(y+t)$, where $s+it \in \mathcal{Q}^{\alpha,a,h}$. Since $\mathcal{Q}^{\alpha,a,h} \subset S^{\alpha,a,h}$ by construction, $0 < s < \epsilon$ and $|t| < Mh\alpha s^p$. This element is in $\mathcal{R}$ if and only if

$$0 < \psi^{-1}(|y|)+s < A \quad \text{and} \quad |y+t| < \psi\big(\psi^{-1}(|y|)+s\big).$$

Now $0 \leqslant \psi^{-1}(|y|) < x \leqslant B$. By our choice of B, we have $0 < \psi^{-1}(|y|)+s < (\epsilon/2)+\epsilon < A$.

Thus, to establish part (1), we must show that $|y+t| < \psi\big(\psi^{-1}(|y|)+s\big)$. As $\mathcal{R}$ is symmetric about the real axis, it suffices to deal with the case $y \geqslant 0, t \geqslant 0$. Notice that ψ satisfies the hypothesis of Lemma 6.2. We have

$$\begin{aligned} \psi\big(\psi^{-1}(|y|)+s\big) &\geqslant y+\psi(s) && \text{(by Lemma 6.2)} \\ &\geqslant y+(1/C)s^p && \text{(by hypothesis).} \end{aligned}$$

Recall that $Mh\alpha < 1/C$. Therefore,

$$|y+t| = y+t < y+(1/C)s^p \leqslant \psi(\psi^{-1}(|y|)+s).$$

This shows that $(\psi^{-1}(|y|)+s)+i(y+t) \in \mathcal{R}$, for $y, t \geqslant 0$. In view of our remark on the symmetry of $\mathcal{R}$, this completes the proof of part (1).

For any $x+iy$ as in the previous paragraphs, $\psi^{-1}(|y|)+o > B \geqslant x$. The first inequality follows from our choice of B. This proves part (2).

Define $K := \{z \in \mathbb{C} \mid o \leqslant \Re(z) \leqslant A,\ |\Im(z)| \leqslant \psi(\Re(z)-o)\}$. Write $\mathcal{R}_B := \{z \in \mathcal{R} \mid \Re(z) \leqslant B\}$. For any $x+iy \in \mathcal{R}_B$:

$$o \leqslant o+\psi^{-1}(|y|) < o+x \leqslant o+B < 2o < A.$$

Furthermore, $|y| = \psi\big((\psi^{-1}(|y|)+o)-o\big)$, whence $o+(\psi^{-1}(|y|)+iy) \in K$. Clearly, K intersects $\{z \in \mathbb{C} \mid \Re(z) = A\}$ and $K \setminus \{z \in \mathbb{C} \mid \Re(z) = A\} \subsetneq \mathcal{R}$. This proves part (3).

Finally, for $x + iy \in \mathcal{R}_B$, $\delta_{\mathcal{R}}(x + iy) \leqslant |(\psi^{-1}(|y|) + iy) - (x + iy)| = |\psi^{-1}(|y|) - x|$ because $\psi^{-1}(|y|) + iy \in \partial\mathcal{R}$. This proves part (4) and completes the proof. □

The next lemma is essentially a parametrized version of the one above. It is related to embedding the model region $\mathcal{Q}^{\alpha,a,h}$ into a caltrop within a spike (see Section 3 to recall terminology), as we shall see in Section 7. A note about our notation: we shall abbreviate $(z_1, \dots, z_{n-1}, z_n) \in \mathbb{C}^n$ as (z', z_n).

Lemma 6.4. *Let $\psi : [0, A] \to [0, +\infty)$ be as in Lemma 6.3. Let*

$$D := \{z \in \mathbb{C}^n \mid 0 < \Re(z_n) < A,\ \Im(z_n)^2 + \|z'\|^2 < \psi(\Re(z_n))^2\}.$$

Let $w' \in \mathbb{C}^{n-1}$ and let

$$\mathcal{R}_{w'} := \pi_n\big[\big((w', 0) + \{0_{n-1}\} \times \mathbb{C}\big) \cap D\big].$$

Write $\alpha = 1/(p - 1)$. Then there exist constants $a, h, B > 0$ and a compact subset K of $\{z \in \mathbb{C}^n \mid \Re(z_n) \leqslant A\}$ that intersects $\{z \in \mathbb{C}^n \mid \Re(z_n) = A\}$ and so that $K \setminus \{z \in \mathbb{C}^n \mid \Re(z_n) = A\} \subsetneq D$, such that for every $w' \in \mathbb{C}^{n-1}$ with $\|w'\| < \psi(B/2)$, and every $\zeta \in \mathcal{R}_{w'}$ with $\Re(\zeta) \leqslant B$, one has

(1) $\big(\psi^{-1}(S(\zeta, w')) + i\Im(\zeta)\big) + \mathcal{Q}^{\alpha,a,h} \subseteq \mathcal{R}_{w'}$;
(2) $\psi^{-1}(S(\zeta, w')) + o > \Re(\zeta)$;
(3) $\big(\psi^{-1}(S(\zeta, w')) + i\Im(\zeta)\big) + o \in \pi_n\big[\big((w', 0) + \{0_{n-1}\} \times \mathbb{C}\big) \cap K\big]$;
(4) $\delta_D((w', \zeta)) \leqslant |\Re(\zeta) - \psi^{-1}\big(S(\zeta, w')\big)|$,

where $S(\zeta, w') := \sqrt{\Im(\zeta)^2 + \|w'\|^2}$ and o is the point in $\mathcal{Q}^{\alpha,a,h}$ given by (6.5).

Remark 6.5. The following expression for $\mathcal{R}_{w'}$ can easily be obtained:

$$\{\zeta \in \mathbb{C} \mid \psi^{-1}(\|w'\|) < \Re(\zeta) < A,\ \Im(\zeta)^2 + \|w'\|^2 < \psi(\Re(\zeta))^2\}.$$

We see that $\mathcal{R}_{w'} \neq \varnothing$ if and only if $\|w'\| < \psi(A)$. In particular, the sets $\mathcal{R}_{w'}$ appearing in the conclusions of the above lemma are non-empty. We also note that $\mathcal{R}_{0_{n-1}}$ is precisely the $\mathcal{R}$ of the last lemma. We shall take the parameters a, h and B, whose existence is asserted above, to be precisely the parameters obtained from the domain $\mathcal{R} = \mathcal{R}_{0_{n-1}}$ using Lemma 6.3 above.

Proof. For simplicity of notation, we shall write $c := 1/C$. The $w' = 0_{n-1}$ case is precisely the content of Lemma 6.3. Let a, h and B be as given by Lemma 6.3. We extract from the proof of Lemma 6.3 a couple of simple facts that follow from this choice of parameters, and which we shall need in this proof:

$$s + it \in \mathcal{Q}^{\alpha,a,h} \Rightarrow\ B + s < 3A/4 < A \text{ and } |t| < cs^p; \tag{6.6}$$

$$o > B. \tag{6.7}$$

We now consider the case $w' \neq 0_{n-1}$. Fix a point $\zeta \in \mathcal{R}_{w'}$, and let $\Re(\zeta) \leqslant B$. That there *is* such a point follows from our bound on $\|w'\|$. An arbitrary element of $(\psi^{-1}(S(\zeta, w')) + i\Im(\zeta)) + \mathcal{Q}^{\alpha,a,h}$ is of the form

$$\big(\psi^{-1}(S(\zeta, w')) + s\big) + i\big(\Im(\zeta) + t\big),$$

where $s + it \in \mathcal{Q}^{\alpha,a,h}$. Such a point belongs to $\mathcal{R}_{w'}$ if and only if

(a) $\psi^{-1}(S(\zeta, w')) + s < A$, and
(b) $\|w'\|^2 + (\Im(\zeta) + t)^2 < \big(\psi(\psi^{-1}(S(\zeta, w')) + s)\big)^2$.

By symmetry, we only need to deal with $\Im(\zeta) \geqslant 0$. We have $\Im(\zeta)^2 + \|w'\|^2 < \psi(\Re(\zeta))^2$. As $\Re(\zeta) \leqslant B$, we have $\Im(\zeta)^2 + \|w'\|^2 < \psi(B)^2$. Therefore

$$\psi^{-1}(S(\zeta, w')) + s < B + s < A.$$

The last inequality follows from (6.6). This verifies (a) above. We now verify (b). We have

$$\psi(\psi^{-1}(S(\zeta, w')) + s) \geqslant S(\zeta, w') + cs^p,$$

by an application of Lemma 6.2. Hence

$$\big(\psi(\psi^{-1}(S(\zeta, w')) + s)\big)^2 - \Im(\zeta)^2 - \|w'\|^2 \geqslant 2cs^p S(\zeta, w') + c^2 s^{2p}.$$

So (b) will follow if we can show that $2\Im(\zeta)t + t^2 < 2cs^p S(\zeta, w') + c^2 s^{2p}$. But this last inequality is obvious in view of (6.6). Thus, (b) is proved, and with it, part (1).

We note that

$$\psi^{-1}\big(S(\zeta, w')\big) + o \geqslant \psi^{-1}(\Im(\zeta)) + o > B \geqslant \Re(\zeta).$$

The second inequality above follows from (6.7). This proves part (2).

Let $K := \{(w', \zeta) \in \mathbb{C}^n \mid o \leqslant \Re(\zeta) \leqslant A,\ S(\zeta, w') \leqslant \psi(\Re(\zeta) - o)\}$. Clearly, K is a compact subset of $\{z \in \mathbb{C}^n \mid \Re(z_n) \leqslant A\}$ that intersects $\{z \in \mathbb{C}^n \mid \Re(z_n) = A\}$, and $K \setminus \{z \in \mathbb{C}^n \mid \Re(z_n) = A\} \subsetneq D$. Fix w' such that $\|w'\| < \psi(B/2)$. Consider a point $\zeta \in \mathcal{R}_{w'}$ such that $\Re(\zeta) \leqslant B$. If we write

$$\eta := \big(\psi^{-1}(S(\zeta, w')) + i\Im(\zeta)\big) + o,$$

then we have

$$\psi^{-1}(\|w'\|) + o \leqslant \Re(\eta) < \Re(\zeta) + o \leqslant B + o < A.$$

This last inequality follows from (6.6) (since $o \in \mathcal{Q}^{\alpha,a,h}$). Furthermore, $S(\eta, w') = S(\zeta, w') = \psi(\Re(\eta) - o)$. Thus, $\eta \in \pi_n\big[\big((w', 0) + \{0_{n-1}\} \times \mathbb{C}\big) \cap K\big]$, which establishes part (3).

As for part (4), if (w', ζ) is as in the last paragraph, then $\psi^{-1}(S(\zeta, w')) + i\Im(\zeta) \in \partial\mathcal{R}_{w'}$. Therefore $\delta_D((w', \zeta)) \leqslant \operatorname{dist}(\zeta, \mathbb{C}\setminus\mathcal{R}_{w'}) \leqslant |\Re(\zeta) - \psi^{-1}(S(\zeta, w'))|$. □

7. Caltrops are visibility domains with respect to the Kobayashi metric

This section is devoted to the proof of Theorem 1.4. Our proof will rely on Theorem 1.5. Recall that, in the discussion related to this theorem, we had mentioned that the utility of Theorem 1.5 lies in that it allows one to identify visibility domains that *do not* possess the Goldilocks property. The concluding paragraphs of this section bear this fact out: we shall show that caltrops are not Goldilocks domains.

We shall need the following basic result:

Lemma 7.1. *Let $\psi : [0, A] \to [0, +\infty)$ denote one of the functions ψ_j occurring in Definition* 1.3. *Then ψ is differentiable at* 0 *and ψ' is continuous on* $[0, A)$, *whence* $\lim_{x\to 0^+} \psi'(x) = 0$.

Proof. That $\psi'(0)$ exists and equals 0 follows simply from the bounds on ψ. Hence, ψ' extends to a function on $[0, A)$. The nature of the discontinuities of the derivative of a univariate function is such that, since ψ' is increasing on $(0, A)$, it cannot have a discontinuity at 0. □

The proof of Theorem 1.4. Since we will need Theorem 1.5 to show that a caltrop $\Omega \subset \mathbb{C}^n$ is a visibility domain with respect to the Kobayashi distance, we will require two different types of estimates. We shall therefore divide our proof into several steps. We begin with the following preliminary remark: if F and G are two non-negative functions that depend on several parameters, then we shall write $F \gtrsim G$ to mean that there exists some constant C that is independent of those parameters such that $G \leqslant C \cdot F$. The expression $F \approx G$ would mean that $F \gtrsim G$ and $G \gtrsim F$.

Step 1. *A lower bound for $\kappa_\Omega(w, \cdot)$ for w contained in a spike*

Given the set of exceptional points $\{q_1, \dots, q_N\} \subset \partial\Omega$, **fix** an exceptional point q_{j^*}. Let $p_{j^*} \in (1, 3/2)$, $\mathsf{U}^{(j^*)} \in U(n)$ and $\psi_{j^*} : [0, A_{j^*}] \to [0, +\infty)$ be the data associated to this exceptional point given by Definition 1.3. Since κ_Ω is invariant under biholomorphisms of Ω, and since the the unitary transformations $\mathsf{U}^{(j)} = \mathsf{U}'_j$ — where U_j, $j = 1, \dots, N$, are the holomorphic maps occurring in Definition 1.3 — preserve the (Euclidean) norms of vectors, we shall, for simplicity of notation, drop the sub/superscript "j^*" from the above-mentioned data and assume without loss of generality that

$$\Omega \cap V_{j^*} = \left\{ z \in \mathbb{C}^n \mid 0 < \Re(z_n) < A,\ \Im(z_n)^2 + \|z'\|^2 < \psi(\Re(z_n))^2 \right\}$$

(so, in the notation just explained, $q_{j^*} = q = 0$).

We shall now construct a negative plurisubharmonic function on Ω that has an explicit form on a substantial portion of $\Omega \cap V_{j^*}$. This will allow us to use Result 2.6 to obtain a lower bound on $\kappa_\Omega(w, \cdot)$ on a portion of $\Omega \cap V_{j^*}$. That there exist such functions does not follow immediately from the existing theory owing to the presence of singularities in $\partial\Omega$. We shall thus construct a function with the desired properties from basic principles. To this end, let us write

$$\rho(z) := \Im(z_n)^2 + \|z'\|^2 - \psi(\Re(z_n))^2 \quad \forall z \in \Omega \cap V_{j^*}.$$

By the Levi-form calculation in Lemma 3.1, by Lemma 7.1, and owing to the properties of ψ, we see that there exists a constant $A' \in (0, A]$ such that

$$\mathscr{L}(\rho)(z; v) \geqslant \|v'\|^2 + 4^{-1}|v_n|^2 \quad \forall (z, v) \in (\Omega \cap V_{j^*}) \times \mathbb{C}^n : 0 < \Re(z_n) < A'. \tag{7.1}$$

Let $U^{(j)}$, $1 \leqslant j \leqslant 4$, be connected open neighbourhoods of 0 (which represents q_{j^*} in our present coordinates) such that:

- $U^{(1)} \Subset U^{(2)} \Subset U^{(3)} \Subset U^{(4)}$;
- $U^{(j)} \cap \Omega = \{z \in \Omega \cap V_{j^*} \mid 0 < \|z\| < jA'/4\}$, $1 \leqslant j \leqslant 4$.

Let $\chi_1 \in \mathcal{C}^\infty(\mathbb{C}^n)$ be such that $\chi_1 : \mathbb{C}^n \to [0, 1]$ and satisfies

$$\chi_1|_{U^{(1)}} \equiv 0, \quad \text{and} \quad \chi_1|_{\mathbb{C}^n \setminus U^{(2)}} \equiv 1.$$

Let ϕ be a smooth, nondecreasing convex function on $[0, +\infty)$ satisfying $\phi(x) = 0$ for each $x \in [0, (A')^2/16]$ that grows very gradually in $((A')^2/16, (A')^2/4]$ and very rapidly in $[9(A')^2/16, +\infty)$ in a manner that we shall specify presently. Set $M_\phi := \sup_{z\in\Omega} \phi(\|z\|^2)$ and write

$$\Phi(z) := \phi(\|z\|^2) - M_\phi \quad \forall z \in \Omega.$$

Clearly, Φ is plurisubharmonic. We compute:

$$\begin{aligned}
\mathscr{L}(\rho + \chi_1\Phi)(z; v) &= \mathscr{L}(\rho)(z; v) + \chi_1(z)\mathscr{L}(\Phi)(z; v) \\
&\quad + 2\Re\Big[\sum_{j,k=1}^{n} \partial_j\chi_1\, \partial_{\overline{k}}\Phi(z) v_j \overline{v}_k\Big] + \Phi(z)\mathscr{L}(\chi_1)(z; v) \\
&\geqslant \|v'\|^2 + 4^{-1}|v_n|^2 - 2\sum_{j,k=1}^{n} \big|\partial_j\chi_1\, \partial_{\overline{k}}\Phi(z)\big|\, |v_j|\, |\overline{v}_k| \\
&\quad - |\Phi(z)|\, |\mathscr{L}(\chi_1)(z; v)| \quad \forall (z, v) \in \big((U^{(2)} \setminus U^{(1)}) \cap \Omega\big) \times \mathbb{C}^n.
\end{aligned}$$

We can drop the term $\chi_1(z)\mathscr{L}(\Phi)(z; v)$ altogether from the right-hand side of the above inequality since it is non-negative. We now state the first of the properties of ϕ alluded to above: ϕ grows so slowly in the interval $((A')^2/16, (A')^2/4]$ that

$$\mathscr{L}(\rho + \chi_1\Phi)(z; v) \geq 2^{-1}\|v'\|^2 + 8^{-1}|v_n|^2 \quad \forall (z, v) \in \big((U^{(2)} \setminus U^{(1)}) \cap \Omega\big) \times \mathbb{C}^n. \tag{7.2}$$

Now pick $\chi_2 \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ such that $\chi_2 : \mathbb{C}^n \to [0, 1]$ and satisfies

$$\chi_2|_{U^{(3)}} \equiv 1, \quad \text{and} \quad \chi_2|_{\mathbb{C}^n \setminus U^{(4)}} \equiv 0.$$

A Levi-form calculation very similar to the one above gives us

$$\begin{aligned}\mathscr{L}(\chi_2\rho+\Phi)(z;v) \geqslant\ & \phi'(\|z\|^2)\|v\|^2+\phi''(\|z\|^2)\,|\langle z,v\rangle|^2\\ & -2\sum_{j,k=1}^{n}\left|\partial_j\chi_2\,\partial_{\overline{k}}\rho(z)\right|\,|v_j|\,|\overline{v}_k|\\ & -|\rho(z)|\,|\mathscr{L}(\chi_2)(z;v)| \quad \forall (z,v)\in\big((U^{(4)}\setminus U^{(3)})\cap\Omega\big)\times\mathbb{C}^n.\end{aligned}$$

The final condition we require on ϕ is that ϕ' becomes so large on $[9(A')^2/16,+\infty)$ that we can find a positive constant $c>0$ so that

$$\mathscr{L}(\chi_2\rho+\Phi)(z;v)\geqslant c\|v\|^2 \quad \forall (z,v)\in\big((U^{(4)}\setminus U^{(3)})\cap\Omega\big)\times\mathbb{C}^n. \tag{7.3}$$

Finally, let us write $u(z) := \chi_1\Phi(z)+\chi_2\rho(z)$ for each $z\in\Omega$. Recall that Φ is plurisubharmonic (which is used in the calculations above). By this fact, and:

- by the maximum principle (applied to Φ), and by the choice of the functions χ_j, $j=1,2$, we see that $u<0$ on Ω;
- from the choice of the functions χ_j, $j=1,2$, and from the inequalities (7.2) and (7.3), it follows that u is plurisubharmonic on Ω.

By the rotational symmetry of the spike $\Omega\cap V_{j^*}$, it follows that for any $w\in\Omega\cap V_{j^*}$ with $\Re(w_n)$ sufficiently small, we have

$$\delta_\Omega(w)=\operatorname{dist}\big(\Re(w_n)+iS(w),\mathsf{graph}(\psi)\big).$$

Here, $S(w)$ is our abbreviation for $\sqrt{\Im(w_n)^2+\|w'\|^2}$. From this last observation and elementary calculus, it follows that for such a w, if $\xi^w\in\partial\Omega$ is a point such that $\|w-\xi^w\|=\delta_\Omega(w)$, then

$$\Re(w_n)-\Re\big(\pi_n(\xi^w)\big)\in\big(0,\,\psi(\Re(w_n))\psi'(\Re(w_n))\big).$$

Thus, it follows from Lemma 7.1 and a few elementary estimates that (we write $\xi_n^w := \pi_n(\xi^w)$ henceforth)

$$\begin{aligned}&\frac{\delta_\Omega(w)}{\psi(\Re(w_n))-S(w)}\\ &=\frac{\left|\big(\Re(\xi_n^w)-\Re(w_n)\big)+i\big(\psi(\Re(\xi_n^w))-S(w)\big)\right|}{\psi(\Re(w_n))-S(w)}\to 1 \text{ as } \Re(w_n)\to 0.\end{aligned}$$

Hence, there exists a constant $A''>0$ such that

$$\{z\in\Omega\cap V_{j^*}\mid 0<\Re(z_n)<A''\}\subset\Omega\cap U^{(1)} \text{ and } \psi(x)\in(0,1)\ \ \forall x\in(0,A''); \tag{7.4}$$

$$\frac{\delta_\Omega(w)}{\psi(\Re(w_n))-S(w)}>\frac{1}{2}\quad \forall w:\Re(w_n)\in(0,A'') \tag{7.5}$$

We now appeal to Result 2.6. Fix a point $w \in \Omega \cap V_{j^*}$ such that $0 < \Re w_n < A''$. Then, there exists a constant $b > 0$ such that

$$\begin{aligned}\kappa_\Omega(w; v) &\geqslant b\frac{\|v\|}{|u(w)|^{1/2}} \qquad (7.6)\\ &= b\,\frac{\|v\|}{\big(\psi(\Re(w_n)) - S(w)\big)^{1/2}\big(\psi(\Re(w_n)) + S(w)\big)^{1/2}}\\ &\quad \text{(by (7.4), given the definitions of } \chi_1, \chi_2)\\ &\geqslant \frac{b}{\sqrt{2}}\,\frac{\|v\|}{\big(\psi(\Re(w_n)) - S(w)\big)^{1/2}} \qquad \text{(by (7.4) above)}\\ &\geqslant \frac{b}{2}\,\frac{\|v\|}{\delta_\Omega(w)^{1/2}} \qquad \text{(by (7.5) above)},\end{aligned}$$

and this estimate holds for any arbitrary w as described above — *i.e.*, $w \in \Omega \cap V_{j^*}$ such that $0 < \Re w_n < A''$.

Step 2. *An upper bound for* M_Ω

Since the exceptional point q_{j^*} in Step 1 was arbitrarily chosen, we actually infer the following from Step 1: there exist a constant $\beta > 0$ and constants $A''_1, \dots, A''_N > 0$ such that

$$\kappa_\Omega(w; v) \geqslant \beta\,\frac{\|v\|}{\delta_\Omega(w)^{1/2}} \quad \forall w \in \Omega \cap \mathsf{U}_j^{-1}\big(\{z \in \mathbb{C}^n \mid \Re(z_n) < A''_j\}\big) \text{ and } \forall v \in \mathbb{C}^n, \qquad (7.7)$$

where $j = 1, \dots, N$. Now, let us write

$$\begin{aligned}\mathcal{M}_0 &:= \partial\Omega \cap \textstyle\bigcap_{1\leqslant j\leqslant N} \mathsf{U}_j^{-1}\big(\{z \in \mathbb{C}^n \mid \Re(z_n) > A''_j/2\}\big),\\ \mathcal{M}_1 &:= \partial\Omega \setminus \textstyle\bigcup_{1\leqslant j\leqslant N} \mathsf{U}_j^{-1}\big(\{z \in \mathbb{C}^n \mid \Re(z_n) < A''_j\}\big).\end{aligned}$$

It is clear from Definition 1.3 and from very standard facts about strongly Levi-pseudoconvex hypersurfaces that the Levi-nondegeneracy condition stated in Result 2.4 holds true at every $\xi \in \mathcal{M}_0$. Thus, it follows from Result 2.4 that there exists an $\overline{\Omega}$-open neighbourhood $\mathcal{V}$ of $\mathcal{M}_1$ and a constant $\beta' > 0$ such that

$$\kappa_\Omega(w; v) \geqslant \beta'\,\frac{\|v\|}{\delta_\Omega(w)^{1/2}} \quad \forall w \in \mathcal{V} \cap \Omega \text{ and } \forall v \in \mathbb{C}^n. \qquad (7.8)$$

Now, by definition, the set

$$\Omega \setminus \Big(\mathcal{V} \cup \textstyle\bigcup_{1\leqslant j\leqslant N} \mathsf{U}_j^{-1}\big(\{z \in \mathbb{C}^n \mid \Re(z_n) < A''_j\}\big)\Big)$$

is compact. Thus, in view of (7.7) and (7.8), it follows that

$$\frac{1}{\kappa_\Omega(w;v)} \lesssim \delta_\Omega(w)^{1/2} \quad \forall w \in \Omega \text{ and } \forall v \in \mathbb{C}^n : \|v\| = 1.$$

In particular, $M_\Omega(r) \lesssim r^{1/2}$.

Step 3. *The behaviour of* k_Ω
Let us initially **fix** an exceptional point in the set $\{q_1, \dots, q_N\}$. Let a_j, h_j and B_j be the constants given by Lemma 6.4 taking $\psi = \psi_j$. For simplicity of notation, we shall denote the first two constants as a and h — with the dependence on j being understood. Consider a point

$$w \in \Omega \cap \mathsf{U}_j^{-1}\big(\{z \in \mathbb{C}^n \mid \Re(z_n) < B_j/2\}\big),$$

and write $\mathsf{U}_j(w) = (\omega', \omega_n)$. Next, consider the holomorphic map $\Psi_{j,w} : \mathcal{Q}^{\alpha,a,h} \to \mathbb{C}^n$ given by

$$\Psi_{j,w}(\zeta) := \mathsf{U}_j^{-1}\big(\omega', \psi_j^{-1}(S(\omega)) + i\Im(\omega_n) + \zeta\big) \quad \forall \zeta \in \mathcal{Q}^{\alpha,a,h},$$

where $S(\omega) := \sqrt{\Im(\omega_n)^2 + \|\omega'\|^2}$, and $\mathcal{Q}^{\alpha,a,h}$ is the domain constructed in Section 6. The parameters a and h are as just described above. We take $\alpha = 1/(p_j - 1)$. Note that this map is a $\mathbb{C}$-affine embedding of $\mathcal{Q}^{\alpha,a,h}$ into $\mathbb{C}^n$. As hinted in Section 6, we shall show that $\Psi_{j,w}$ embeds $\mathcal{Q}^{\alpha,a,h}$ in Ω — by which we can estimate k_Ω.

Observe that $\Re(\omega_n) < B_j/2$. Hence, given the set to which w belongs and by Definition 1.3, $\|\omega'\| < \psi_j(B_j/2)$. Thus, it follows from parts (1) and (2) of Lemma 6.4 that:

(a) With w as chosen above,

$$\begin{aligned}&\{\omega'\} \times \big(\psi_j^{-1}(S(\omega)) + i\Im(\omega_n) + \mathcal{Q}^{\alpha,a,h}\big)\\ &\subset \big\{(z', z_n) \in \mathbb{C}^n \mid \Re(z_n) \in (0, A_j),\ \Im(z_n)^2 + \|z'\|^2 < \psi_j\big(\Re(z_n)\big)^2\big\};\end{aligned}$$

(b) $\Re(\omega_n) \in (\psi_j^{-1}(S(\omega)),\ \psi_j^{-1}(S(\omega)) + o)$.

Here $o \in \mathcal{Q}^{\alpha,a,h}$ is as provided by (6.5) for the above-mentioned choice of parameters. Now, write

$$K_j := \mathsf{U}_j^{-1}(K) \quad \text{and} \quad z_w := \Psi_{j,w}(o),$$

where K is the compact set given by Lemma 6.4 taking $\psi = \psi_j$ in that lemma. Then, it follows from part (3) of the latter lemma that

$$z_w \in K_j \quad \forall w \in \Omega \cap V_j : 0 < \Re(\omega_n) < B_j/2. \tag{7.9}$$

From (a) we see that $\Psi_{j,w}(\mathcal{Q}^{\alpha,a,h}) \subset \Omega$. By definition, $\Psi_{j,w}(-\psi_j^{-1}(S(\omega)) + \Re(\omega_n)) = w$. Thus, as holomorphic maps are contractive relative to the Kobayashi distance, we have

$$\mathsf{k}_\Omega(z_w, w) \leqslant \mathsf{k}_{\mathcal{Q}^{\alpha,a,h}}(o, -\psi_j^{-1}(S(\omega)) + \Re(\omega_n)).$$

In view of (b), part (3) of Proposition 6.1 gives us — taking $x_0 = o$ in that proposition — the estimate

$$\mathsf{k}_\Omega(z_w, w) \leqslant C^{(j)} + \frac{\pi}{4h} |\Re(\omega_n) - \psi_j^{-1}(S(\omega))|^{-(p_j-1)},$$

for some constant $C^{(j)} > 0$. Since the maps U_j and U_j^{-1} preserve Euclidean distances, the above inequality together with part (4) of Lemma 6.4 gives us the following:

$$\mathsf{k}_\Omega(z_w,w) \leqslant C^{(j)} + \frac{\pi}{4h}\delta_\Omega(w)^{-(p_j-1)} \quad \forall w \in \Omega \cap V_j : 0 < \Re(\omega_n) < B_j/2. \tag{7.10}$$

Since the exceptional point q_j was chosen arbitrarily in this discussion, the statements (7.9) and (7.10) hold for each $j = 1, \dots, N$.

Since $\partial\Omega$ is of class $\mathcal{C}^2$ away from the points $q_1, \dots, q_N$, and Ω is bounded, it is routine to find a compact set $K_0 \subset \Omega$ and a constant $R > 0$ such that for each point

$$w \in \Omega \setminus \Big(K_0 \cup \bigcup\nolimits_{1\leqslant j\leqslant N} \mathsf{U}_j^{-1}\big(\{z \in \mathbb{C}^n \mid \Re(z_n) < B_j/2\}\big)\Big) \tag{7.11}$$

there exists a point

$$\xi^w \in \partial\Omega \setminus \bigcup\nolimits_{1\leqslant j\leqslant N} \mathsf{U}_j^{-1}\big(\{z \in \mathbb{C}^n \mid \Re(z_n) < B_j/4\}\big)$$

so that, if η^w denotes the unit inward-pointing normal vector to $\partial\Omega$, then

(a′) $\xi^w + D(R; R)\eta^w \subset \Omega$;
(b′) w lies on the line segment joining ξ^w to $\xi^w + R\eta^w =: z^w$; and
(c′) $z^w \in K_0$.

Thus, for each w as indicated above, there is a unique number $t(w) \in (0, R)$ such that $\xi^w + t(w)\eta^w = w$. From this (and the fact that holomorphic maps are contractive relative to the Kobayashi distance) it follows that

$$\begin{aligned}
\mathsf{k}_\Omega(z^w, w) \leqslant \mathsf{k}_{D(R;R)}(0, t(w)) &= \frac{1}{2}\log\left(\frac{2 - (t(w)/R)}{t(w)/R}\right) \\
&\leqslant \log(\sqrt{2}) + \frac{1}{2}\log\left(\frac{1}{\|\xi^w - w\|}\right) \\
&\leqslant \log(\sqrt{2}) + \frac{1}{2}\log\left(\frac{1}{\delta_\Omega(w)}\right) \\
&\quad \forall w \text{ satisfying the condition given by (7.11).}
\end{aligned} \tag{7.12}$$

Let us now fix a point $z_0 \in \Omega$. Write

$$K^* := K_0 \cup K_1 \cup \cdots \cup K_N,$$
$$C_0 := \sup_{x \in K^*} \mathsf{k}_\Omega(z_0, x) + \max\big(\log(\sqrt{2}), C^{(1)}, \dots, C^{(N)}\big).$$

Then, by the triangle inequality for k_Ω, (7.9), and by the inequalities (7.10) and (7.12) it follows that there exists a constant $C_1 > 0$ such that

$$\mathsf{k}_\Omega(z_0, z) \leqslant C_0 + C_1 \delta_\Omega(z)^{-\max_{1 \leqslant j \leqslant N} p_j + 1} \quad \forall z \in \Omega.$$

Step 4. *Caltrops are visibility domains with respect to the Kobayashi distance*
Let us write $p_0 := \max_{1 \leqslant j \leqslant N} p_j$. Then, by hypothesis, $p_0 \in (1, 3/2)$. We shall complete the proof using Theorem 1.5. In the notation of that theorem, we can — using the conclusion of Step 3 — take $f(r) = C_0 + C_1 r^{p_0 - 1}$. Thus, using the conclusion of Step 2, we have

$$0 \leqslant \frac{M_\Omega(r)}{r^2} f'\!\left(\frac{1}{r}\right) \lesssim \frac{r^{1/2}}{r^2} \cdot r^{2-p_0} = \frac{1}{r^{p_0 - (1/2)}}.$$

As $p_0 < 3/2$, we have

$$0 \leqslant \int_0^{r_0} \frac{M_\Omega(r)}{r^2} f'\!\left(\frac{1}{r}\right) dr \lesssim \int_0^{r_0} \frac{dr}{r^{p_0 - (1/2)}} < \infty$$

for r_0 so small that $(0, r_0)$ is included in the domain of the integrand. Hence, we conclude from Theorem 1.5 that Ω is a visibility domain with respect to the Kobayashi distance.

Step 5. *Caltrops are not Goldilocks domains*
We will show that the condition on the growth of the Kobayashi distance that Goldilocks domains must satisfy fails in a caltrop. To do so, we fix an exceptional point q_{j^*} and refer the reader to Step 1 for an explanation for why we can, without loss of generality, take $\Omega \cap V_{j^*}$ to be

$$\Omega \cap V_{j^*} = \big\{z \in \mathbb{C}^n \mid 0 < \Re(z_n) < A,\ \Im(z_n)^2 + \|z'\|^2 < \psi(\Re(z_n))^2\big\} \tag{7.13}$$

(recall that V_{j^*} is the neighbourhood of q_{j^*} given by Definition 1.3). As in Step 1, we drop, for the moment, the sub/superscript "j^*".

At this stage, we shall need the following result:

Lemma 7.2. *Fix an exceptional point $q_{j^*} \in \partial\Omega$ and let $(z_1, \dots, z_n)$ be the system of holomorphic coordinates centred at q_{j^*} such that $\Omega \cap V_{j^*}$ has the form* (7.13). *Let A'' be as introduced just prior to* (7.4)*, and let $z_0 = (0, \dots, 0, A''/2)$. Then, for any $z \in \Omega \cap V_{j^*}$ such that* $0 < \Re(z_n) < A''/2$, *we have*

$$\mathsf{k}_\Omega(z_0, z) \gtrsim \Re(z_n)^{-(p-1)} - (A''/2)^{-(p-1)}. \tag{7.14}$$

We shall defer the proof of this lemma until the end of this section. Instead, let us use it to complete this proof. Write $\mathfrak{z}^x := (0, \dots, 0, x), 0 < x < A''/2$. Now, take $z = \mathfrak{z}^x$ in the above lemma to get

$$\mathsf{k}_\Omega(z_0, \mathfrak{z}^x) \gtrsim x^{-(p-1)} - (A''/2)^{-(p-1)}.$$

Now, substitute $\mathfrak{z}^x$ for the w in the statement just prior to (7.4) in Step 1 to infer that $\delta_\Omega(\mathfrak{z}^x) \approx x^p$ for any $x \in (0, A''/2)$. Applying this to the last estimate, we have

$$\mathsf{k}_\Omega(z_0, \mathfrak{z}^x) \gtrsim \delta_\Omega(\mathfrak{z}^x)^{-1+(1/p)} - (A''/2)^{-(p-1)}.$$

Since, $p = p_{j^*} > 1$, $\delta_\Omega(\mathfrak{z}^x)^{-1+(1/p)}/\log\big(1/\delta_\Omega(\mathfrak{z}^x)\big) \to +\infty$ as $\mathfrak{z}^x \to q_{j^*}$. Thus $\mathsf{k}_\Omega(z_0, \mathfrak{z}^x)$ **cannot** satisfy the upper bound (1.1) for any choice of constants $C, \alpha > 0$ as $\mathfrak{z}^x \to q_{j^*}$. Thus the caltrop Ω is not a Goldilocks domain. ☐

We now provide the following:

Proof of Lemma 7.2. Fix a $z \in \Omega \cap V_{j^*}$ with $0 < \Re(z_n) < A''/2$. In proving this lemma, we shall use a slightly different lower bound for κ_Ω, which was also derived in Step 1. Let

$$\mathscr{C}(z) := \text{the class of all piecewise } \mathcal{C}^1 \text{ paths } \gamma : ([0,1], 0, 1) \to (\Omega, z_0, z).$$

As discussed at the beginning of Section 2:

$$\mathsf{k}_\Omega(z_0, z) = \inf_{\gamma \in \mathscr{C}(z)} \int_0^1 \kappa_\Omega(\gamma(t); \gamma'(t))\, dt. \tag{7.15}$$

Pick a $\gamma \in \mathscr{C}(z)$. Since $\Re(\gamma_n)$ is a continuous function and $\Re(\gamma_n)([0,1]) \supset [\Re(z_n), A''/2]$, it follows from elementary topological considerations that there exist numbers $\alpha, \beta \in [0,1]$ such that

$$\Re(\gamma_n)([\alpha, \beta]) = [\Re(z_n), A''/2].$$

Therefore

$$\begin{aligned}\int_0^1 \kappa_\Omega(\gamma(t); \gamma'(t))\, dt &\geqslant \int_\alpha^\beta \kappa_\Omega(\gamma(t); \gamma'(t))\, dt \\ &\geqslant \int_\alpha^\beta \frac{b\|\gamma'(t)\|}{|u(\gamma(t))|^{1/2}}\, dt,\end{aligned} \tag{7.16}$$

The second inequality above follows from the estimate (7.6).

For any point $w \in \Omega \cap V_{j^*}$ with $0 < \Re(w_n) < A''$ we have

$$|u(w)| = \psi(\Re(w_n))^2 - \|w'\|^2 - \Im(w_n)^2 \leqslant \psi(\Re(w_n))^2 \leqslant C^2 \Re(w_n)^{2p}$$

where $C > 0$ is the constant C_{j^*} mentioned in Definition 1.3. Therefore, (7.16) gives us (the last three integrals below are Riemann integrals; it is not hard to establish that the integrands are Riemann integrable):

$$\begin{aligned}\int_0^1 \kappa_\Omega(\gamma(t); \gamma'(t))\, dt &\geqslant \frac{b}{C}\int_\alpha^\beta \frac{|(\Re(\gamma_n))'(t)|}{\Re(\gamma_n(t))^p}\, dt \\ &\geqslant \frac{b}{C}\left|\int_\alpha^\beta \frac{(\Re(\gamma_n))'(t)}{\Re(\gamma_n(t))^p}\, dt\right| \\ &= \frac{b}{C}\int_{\Re(z_n)}^{A''/2} \frac{1}{t^p}\, dt\,.\end{aligned}$$

A few words about the change-of-variables formula that gives the last equality: since γ is piecewise $\mathcal{C}^1$, we invoke (a small refinement of) the classical change-of-variables formula on a finite collection of subintervals that tile $[\alpha, \beta]$. Recalling that γ was chosen arbitrarily from the class $\mathscr{C}(z)$, this last estimate, together with (7.15), gives us (7.14). ☐

8. Caltrops are taut

In this section we shall prove that any caltrop is taut. While this is believable, it takes a little effort to show owing to the exceptional points in the boundary of a caltrop Ω. At these points, $\partial\Omega$ is not just non-smooth but is not even Lipschitz (were $\partial\Omega$ Lipschitz, tautness would have followed from a result of Kerzman–Rosay [21]). We first need the following standard result.

Lemma 8.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ and suppose that for each $z_0 \in \Omega$ and $\xi \in \partial\Omega$, we have*

$$\lim_{\Omega \ni w \to \xi} \mathsf{k}_\Omega(z_0, w) = +\infty. \tag{8.1}$$

Then the metric space $(\Omega, \mathsf{k}_\Omega)$ is (Cauchy) complete.

The proof of this lemma involves very standard arguments. We use the conclusion of Result 2.1 and the fact that the metric-topology on Ω induced by k_Ω coincides with its standard topology. We skip the routine details.

With this, we are in a position to prove the following:

Theorem 8.2. *Caltrops are complete relative to the Kobayashi distance. In particular, they are taut.*

Proof. Let Ω be a caltrop, and let $\{q_1, \dots, q_N\} \subset \partial\Omega$ be the set of exceptional boundary points. Fix a point $z_0 \in \Omega$ and a point $\xi \in \partial\Omega$. First, we consider the case where $\xi \in \partial\Omega\setminus\{q_1, \dots, q_N\}$. Pick a point $\eta \in \partial\Omega\setminus(\{q_1, \dots, q_N\}\cup\{\xi\})$. Thus, $\partial\Omega$ is strongly Levi-pseudoconvex around ξ and η. We now appeal to Result 2.3:

let V_ξ and V_η be the neighbourhoods and let $C > 0$ be the constant given by this result. Let b_η be some point in $\Omega \cap V_\eta$. Consider any sequence $(w_\nu)_{\nu \geqslant 1} \subset \Omega$ such that $w_\nu \to \xi$. Without loss of generality, we may assume that this sequence is contained in $\Omega \cap V_\xi$. Then, Result 2.3 tells us that

$$\begin{aligned} \mathsf{k}_\Omega(z_0, w_\nu) &\geqslant \mathsf{k}_\Omega(w_\nu, b_\eta) - \mathsf{k}_\Omega(b_\eta, z_0) \\ &\geqslant 2^{-1} \log \frac{1}{\delta_\Omega(w_\nu)} + 2^{-1} \log \frac{1}{\delta_\Omega(b_\eta)} - \mathsf{k}_\Omega(b_\eta, z_0) - C \\ &\to +\infty \text{ as } \nu \to \infty. \end{aligned}$$

As ξ was arbitrarily chosen from $\partial\Omega \setminus \{q_1, \dots, q_N\}$, the above establishes (8.1) for any non-exceptional boundary point.

Now, let ξ be an exceptional boundary point. As in the statement of Lemma 7.2, call this point q_{j^*} and let $(z_1, \dots, z_n)$ be the system of holomorphic coordinates described in this lemma. Consider any sequence $(w_\nu)_{\nu \geqslant 1} \subset \Omega$ such that $w_\nu \to q_{j^*}$. Without loss of generality, we may assume that this sequence is contained in $\Omega \cap V_{j^*}$. Now, Lemma 7.2 is stated keeping in mind a specific assumption about $\Omega \cap V_{j^*}$ (stated just prior to it). Here too we may without loss of generality assume that $\Omega \cap V_{j^*}$ **is** the set given by (7.13). This is because the coordinates $(z_1, \dots, z_n)$ are given by a biholomorphism defined on all of Ω (indeed, on all of $\mathbb{C}^n$). With this assumption, we shall identify $z_n(w_\nu)$ and $\pi_n(w_\nu) =: w_{\nu,n}$. Then (with this assumption) we have

$$\Re(w_{\nu,n}) \to 0 \text{ as } \nu \to \infty. \tag{8.2}$$

Let A'' the constant given by Lemma 7.2, and let us denote the point z_0 mentioned in this lemma by ζ_{j^*} (to avoid confusion with the z_0 fixed above). Then, this lemma tells us that

$$\mathsf{k}_\Omega(\zeta_{j^*}, w_\nu) \gtrsim \Re(w_{\nu,n})^{-(p-1)} - (A''/2)^{-(p-1)} \to +\infty \text{ as } \nu \to \infty.$$

The last statement follows from (8.2). Owing to the triangle inequality for k_Ω, the above suffices to establish (8.1) for $\xi = q_{j^*}$. Together with the conclusion of the previous paragraph, we conclude — using Lemma 8.1 — that $(\Omega, \mathsf{k}_\Omega)$ is (Cauchy) complete.

As $(\Omega, \mathsf{k}_\Omega)$ is complete, it follows from a result of Kiernan [22] that Ω is taut. □

9. Wolff–Denjoy theorems

We now have all the tools needed to prove the two Wolff–Denjoy-type theorems, and a corollary, stated in Section 1. We reiterate that the key heuristic in the proof of Theorem 1.8 is as stated in the second paragraph following the statement of Theorem 1.8. Since that heuristic is entirely a consequence of visibility, large parts of the proof below will be similar to the proof of [9, Theorem 1.10] for Goldilocks domains, which also relies on this heuristic. The supporting lemmas/theorems to the proof below are those that show that the quantitative conditions defining a Goldilocks domain are not needed.

The proof of Theorem 1.8 involves the analysis of two separate cases, one of which is rather technical. This is because we do **not** assume that $(\Omega, \mathsf{k}_\Omega)$ is Cauchy complete in Theorem 1.8 — to do so would be too restrictive. To illustrate: it is not known whether, for a weakly pseudoconvex domain $\Omega \Subset \mathbb{C}^n$, $n \geq 3$, $(\Omega, \mathsf{k}_\Omega)$ is Cauchy complete (that such a domain is a visibility domain follows from [9, Theorem 1.4]). In contrast, tautness is much simpler to determine in practice, and suffices for the conclusion of Theorem 1.8. With these words, we give the following:

Proof of Theorem 1.8. Since Ω is taut, it follows from a result by Abate [2, Theorem 2.4.3] that either the set $\{F^\nu \mid \nu \in \mathbb{Z}_+\}$ is relatively compact in $\mathcal{O}(\Omega; \Omega)$ or $(F^\nu)_{\nu\geqslant 1}$ is compactly divergent on Ω. In the former case, clearly, for each $z \in \Omega$, the orbit $\{F^\nu(z) \mid \nu \in \mathbb{Z}_+\}$ is relatively compact in Ω.

Hence we now suppose that $(F^\nu)_{\nu\geqslant 1}$ is compactly divergent. By Montel's theorem, there exist subsequences of $(F^\nu)_{\nu\geqslant 1}$ that converge uniformly on compact subsets of Ω to $\partial\Omega$-valued holomorphic maps. By Theorem 4.3, the latter maps are constant maps. Thus, we shall identify the set

$$\Gamma := \overline{\{F^\nu \mid \nu \in \mathbb{Z}_+\}}^{\text{compact-open}} \setminus \{F^\nu \mid \nu \in \mathbb{Z}_+\}$$

as a set of points in $\partial\Omega$. In a similar vein, we shall refer to the constant maps const_p, where $p \in \partial\Omega$, simply as p. Our goal is to show that Γ is a single point. We assume, to get a contradiction, that Γ contains at least two points. We divide our discussion into two cases:

Case 1. We first consider the case in which for some (and hence any) $o \in \Omega$,

$$\limsup_{\nu\to\infty} \mathsf{k}_\Omega(F^\nu(o), o) = \infty.$$

We ought to mention here that (as implied by the discussion following the statement of Theorem 1.8) the essence of the argument under the heading "Case 1" in the proof of [9, Theorem 1.10] applies in the present, more general, setting. The chief differences are as follows:

- The lemmas/propositions supporting the two arguments differ;
- Since the Goldilocks condition in [9] involves an upper bound on k_Ω, certain inequalities and observations (*e.g.*, (9.2) below) needed no argument in the latter work, but for which we provide explanations (when needed) here.

In this case we can find a strictly increasing sequence $(\nu_i)_{i\geqslant 1} \subset \mathbb{Z}_+$ such that for every $i \in \mathbb{Z}_+$ and every $k \leqslant \nu_i$, $\mathsf{k}_\Omega(F^k(o), o) \leqslant \mathsf{k}_\Omega(F^{\nu_i}(o), o)$. By passing to a subsequence and relabelling, if necessary, we may assume that $F^{\nu_i} \to \xi$ uniformly on compact subsets of Ω for some $\xi \in \partial\Omega$. By assumption, there is a subsequence $(F^{\mu_j})_{j\geqslant 1}$ that converges uniformly on compact subsets to η, where $\eta \in \partial\Omega$ and $\eta \neq \xi$. By Proposition 4.1, it cannot be the case that

$$\limsup_{j\to\infty} \mathsf{k}_\Omega(F^{\mu_j}(o), o) = \infty,$$

since $\eta \neq \xi$. Therefore, $\limsup_{j\to\infty} \mathsf{k}_\Omega(F^{\mu_j}(o), o) < \infty$. Hence, by the triangle inequality:

$$\begin{aligned}
&\limsup_{i\to\infty}\limsup_{j\to\infty} \mathsf{k}_\Omega\big(F^{\nu_i}(o), F^{\mu_j}(o)\big) \\
&\geqslant \limsup_{i\to\infty}\limsup_{j\to\infty} \big(\mathsf{k}_\Omega\big(F^{\nu_i}(o), o\big) - \mathsf{k}_\Omega\big(F^{\mu_j}(o), o\big)\big) \\
&\geqslant \limsup_{i\to\infty}\Big[\mathsf{k}_\Omega\big(F^{\nu_i}(o), o\big) - \limsup_{j\to\infty}\mathsf{k}_\Omega\big(F^{\mu_j}(o), o\big)\Big] \\
&= \infty.
\end{aligned} \tag{9.1}$$

Fix an $\ell \in \mathbb{Z}_+$. When we apply Theorem 4.3 to any subsequence of $(F^{\mu_j-\ell})_{j\geqslant 1}$ that converges uniformly on compact subsets of Ω, we get — since any such subsequence converges to η on the compact $K_\ell := \{F^\ell(o)\}$ — that

$$(F^{\mu_j-\ell})_{j\geqslant 1} \text{ converges uniformly on compact subsets to } \eta. \tag{9.2}$$

Let us define

$$M_\ell := \limsup_{j\to\infty} \mathsf{k}_\Omega(F^{\mu_j-\ell}(o), o).$$

We claim that

$$\limsup_{\ell\to\infty} M_\ell < \infty.$$

Suppose not. Then there is a strictly increasing sequence $(\ell_k)_{k\geqslant 1} \subset \mathbb{Z}_+$ such that for each $k \in \mathbb{Z}_+$, $M_{\ell_k} > k$. Next, we can choose positive integers $j_1 < j_2 < j_3 < \dots$ such that for each $k \in \mathbb{Z}_+$

$$\|F^{\mu_{j_k}-\ell_k}(o) - \eta\| < 1/k \quad \text{and} \quad \mathsf{k}_\Omega\big(F^{\mu_{j_k}-\ell_k}(o), o\big) > k.$$

These inequalities imply that $F^{\mu_{j_k}-\ell_k}(o) \to \eta$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \mathsf{k}_\Omega\big(F^{\mu_{j_k}-\ell_k}(o), o\big) = \infty,$$

which contradicts Proposition 4.1, since $\eta \neq \xi$. Hence $\limsup_{\ell\to\infty} M_\ell < +\infty$, as claimed. Then

$$\begin{aligned}
\limsup_{i\to\infty}\limsup_{j\to\infty} \mathsf{k}_\Omega\big(F^{\nu_i}(o), F^{\mu_j}(o)\big) &\leqslant \limsup_{i\to\infty}\limsup_{j\to\infty} \mathsf{k}_\Omega\big(o, F^{\mu_j-\nu_i}(o)\big) \\
&= \limsup_{i\to\infty} M_{\nu_i} < \infty.
\end{aligned}$$

This contradicts (9.1), and finishes the consideration of Case 1.

Case 2. We now consider the case in which for some (and hence any) $o \in \Omega$,

$$\limsup_{\nu\to\infty} \mathsf{k}_\Omega(F^\nu(o), o) < \infty.$$

The argument that follows is almost identical to that under the heading "Case 2" in the proof of [9, Theorem 1.10]. However, since the argument is rather technical, we reproduce it below instead of directing the reader elsewhere. Recall that, by assumption, there exist two distinct points $\xi, \eta \in \Gamma$. We choose strictly increasing sequences $(\nu_i)_{i\geqslant 1}, (\mu_j)_{j\geqslant 1} \subset \mathbb{Z}_+$ such that $F^{\nu_i} \to \xi$ and $F^{\mu_j} \to \eta$ uniformly on compact subsets of Ω. Choose $\overline{\Omega}$-open neighbourhoods V_ξ and V_η of ξ and η, respectively, such that $\overline{V_\xi} \cap \overline{V_\eta} = \varnothing$. By the fact that Ω is a visibility domain with respect to the Kobayashi distance, there exists a compact subset K of Ω such that for every $(1, 1)$-almost-geodesic $\sigma : [0, T] \to \Omega$ satisfying $\sigma(0) \in V_\xi$ and $\sigma(T) \in V_\eta$, $\mathsf{range}(\sigma) \cap K \neq \varnothing$.

Next, for $\delta > 0$ arbitrary, we define $G_\delta : K \times K \to [0, +\infty)$ by

$$G_\delta(x_1, x_2) := \inf\{\mathsf{k}_\Omega(F^m(x_1), x_2) \mid m \in \mathbb{Z}_+ \text{ and } \|\xi - F^m(x_1)\| < \delta\}.$$

By the hypothesis of Case 2, $\sup_{\delta>0;\ x_1, x_2\in K} G_\delta(x_1, x_2) < \infty$. Fix $x_1, x_2 \in K$; if $0 < \delta_1 < \delta_2$, then $G_{\delta_1}(x_1, x_2) \geqslant G_{\delta_2}(x_1, x_2)$. So, for any $x_1, x_2 \in K$,

$$G(x_1, x_2) := \lim_{\delta\to 0^+} G_\delta(x_1, x_2)$$

is well-defined. We also define

$$\epsilon := \liminf_{z\to\eta} \inf_{y\in K} \mathsf{k}_\Omega(y, z).$$

Note that by Result 2.1, $\epsilon > 0$. Choose points $q_1, q_2 \in K$ such that

$$G(q_1, q_2) < \inf\{G(x_1, x_2) \mid x_1, x_2 \in K\} + \epsilon.$$

By an argument similar to the one leading to (9.2), for each fixed $j \in \mathbb{Z}_+$

$$(F^{\nu_i+\mu_j})_{i\geqslant 1} \text{ converges uniformly on compact subsets to } \xi.$$

Therefore, we can find a strictly increasing sequence $(i_j)_{j\geqslant 1} \subset \mathbb{Z}_+$ such that:

(a) $(F^{\nu_{i_j}})_{j\geqslant 1}$ converges uniformly on compact subsets to ξ;
(b) $(F^{\nu_{i_j}+\mu_j})_{j\geqslant 1}$ converges uniformly on compact subsets to ξ;
(c) $\lim_{j\to\infty} \mathsf{k}_\Omega(F^{\nu_{i_j}}(q_1), q_2) = G(q_1, q_2)$.

Finally, choose a sequence $(\kappa_j)_{j\geqslant 1}$ such that $0 < \kappa_j \leqslant 1$ for all j and such that $\kappa_j \to 0^+$ as $j \to \infty$. By Proposition 4.4 of [9] — which guarantees the existence of (λ, κ)-almost-geodesics joining a given pair of points, for any $\lambda \geqslant 1$ and $\kappa > 0$ — for each j there exists a $(1, \kappa_j)$-almost-geodesic $\sigma_j : [0, T_j] \to \Omega$ such that $\sigma_j(0) = F^{\nu_{i_j}+\mu_j}(q_1)$ and $\sigma_j(T_j) = F^{\mu_j}(q_2)$. Clearly, for sufficiently large j, $\sigma_j(0) \in V_\xi$ and $\sigma_j(T_j) \in V_\eta$. Now, because every σ_j is a $(1, 1)$-almost-geodesic,

$\mathsf{range}(\sigma_j) \cap K \neq \varnothing$ for each j. Hence, for each j, we may choose a point $x_j^* \in \mathsf{range}(\sigma_j) \cap K$. Since K is compact, we may, by passing to a subsequence and relabelling, assume that $x_j^* \to x^0 \in K$ as $j \to \infty$. Therefore, by Lemma 2.9, we have

$$\mathsf{k}_\Omega(F^{\nu_{i_j}+\mu_j}(q_1), F^{\mu_j}(q_2)) \geqslant \mathsf{k}_\Omega(F^{\nu_{i_j}+\mu_j}(q_1), x_j^*) + \mathsf{k}_\Omega(x_j^*, F^{\mu_j}(q_2)) - 3\kappa_j. \quad (9.3)$$

Now

$$\begin{aligned}\liminf_{j\to\infty} \mathsf{k}_\Omega(F^{\nu_{i_j}+\mu_j}(q_1), x_j^*) &\geqslant \liminf_{j\to\infty} \big(\mathsf{k}_\Omega(F^{\nu_{i_j}+\mu_j}(q_1), x^0) - \mathsf{k}_\Omega(x^0, x_j^*)\big) \\ &= \liminf_{j\to\infty} \mathsf{k}_\Omega(F^{\nu_{i_j}+\mu_j}(q_1), x^0) - \lim_{j\to\infty} \mathsf{k}_\Omega(x^0, x_j^*) \quad (9.4) \\ &= \liminf_{j\to\infty} \mathsf{k}_\Omega(F^{\nu_{i_j}+\mu_j}(q_1), x^0) \geqslant G(q_1, x^0).\end{aligned}$$

Again, from the definition of ϵ,

$$\liminf_{j\to\infty} \mathsf{k}_\Omega(x_j^*, F^{\mu_j}(q_2)) \geqslant \epsilon.$$

Therefore from (9.3) we obtain, since $\lim_{j\to\infty} \kappa_j = 0$:

$$\begin{aligned}\liminf_{j\to\infty} \mathsf{k}_\Omega(F^{\nu_{i_j}+\mu_j}(q_1), F^{\mu_j}(q_2)) &\geqslant \liminf_{j\to\infty} \mathsf{k}_\Omega(F^{\nu_{i_j}+\mu_j}(q_1), x_j^*) \\ &\quad + \liminf_{j\to\infty} \mathsf{k}_\Omega(x_j^*, F^{\mu_j}(q_2)) \\ &\geqslant G(q_1, x^0) + \epsilon. \qquad \text{(by (9.4) above)}\end{aligned}$$

On the other hand,

$$\limsup_{j\to\infty} \mathsf{k}_\Omega(F^{\nu_{i_j}+\mu_j}(q_1), F^{\mu_j}(q_2)) \leqslant \limsup_{j\to\infty} \mathsf{k}_\Omega(F^{\nu_{i_j}}(q_1), q_2) = G(q_1, q_2).$$

Recall that the sequence $(i_j)_{j\geqslant 1}$ has been so picked that the last equality holds true. From the last two inequalities we obtain

$$G(q_1, q_2) \geqslant G(q_1, x^0) + \epsilon,$$

which is a contradiction to the choice of q_1 and q_2. This finishes the consideration of Case 2.

The above arguments show that we have a contradiction in either case, whence our assumption about Γ must be wrong. This completes the proof of the theorem. □

The conclusions of the last theorem constitute a step in the following:

The proof of Theorem 1.9. Since Ω is a taut bounded domain, Result 2.2-(2.2) tells us that it is pseudoconvex. It is well known — see Theorem 4.2.7 of [17], for instance — that $H^j(\Omega; \mathbb{C}) = 0$ for all $j > n$ since Ω is pseudoconvex. By the universal coefficient theorem,

$$\dim_{\mathbb{C}}(H^j(\Omega;\mathbb{C})) = \dim_{\mathbb{Q}}(H^j(\Omega;\mathbb{Q})) = \mathrm{rank}(H_j(\Omega;\mathbb{Z})) \quad \forall j \in \mathbb{N}.$$

From the last two statements, together with our hypothesis, it follows that

$$H^j(\Omega;\mathbb{Q}) = 0 \quad \forall j \in \mathbb{N},\ j \text{ odd},$$
$$\dim_{\mathbb{Q}}(H^j(\Omega;\mathbb{Q})) < \infty \quad \forall j \in \mathbb{N},\ j \text{ even}.$$

Therefore we can invoke Corollary 2.10 from the article [3] by Abate to conclude that either F has a periodic point in Ω or $(F^\nu)_{\nu\geqslant 1}$ is compactly divergent. In the latter case, the first outcome of the dichotomy presented by Theorem 1.8 cannot hold true. Thus, by Theorem 1.8, there exists a $\xi \in \partial\Omega$ such that $(F^\nu)_{\nu\geqslant 1}$ converges uniformly on compact subsets of Ω to const_ξ. However, this conclusion is not possible if F has a periodic point in Ω. So, in this case, the dichotomy presented by Theorem 1.8 implies that for each $z \in \Omega$, the orbit $\{F^\nu(z) \mid \nu \in \mathbb{Z}_+\}$ is relatively compact in Ω. This completes the proof of the theorem. □

We are finally in a position to give a proof of Corollary 1.10. The phrase "finite type" refers to the finiteness of the D'Angelo 1-type. We shall not define this term here: we refer the reader to [12] for a definition.

The proof of Corollary 1.10. If Ω is a bounded pseudoconvex domain of finite type, then, by definition, $\partial\Omega$ is at least $\mathcal{C}^2$-smooth. Therefore, it follows from a theorem of Kerzman–Rosay [21, Proposition 2] that Ω is taut. On the other hand, if Ω is a caltrop, then we have shown — see Theorem 8.2 above — that Ω is taut.

If Ω is a bounded pseudoconvex domain of finite type, then — in the terminology of [9] — it is a Goldilocks domain and thus a visibility domain; see Section 2 and Theorem 1.4 of [9]. If Ω is a caltrop, then Theorem 1.4 tells us that it is a visibility domain. Thus, in either case, Ω satisfies all the conditions of Theorem 1.9. Hence, the corollary follows. □

References

[1] M. ABATE, *Horospheres and iterates of holomorphic maps*, Math. Z. **198** (1988), 225–238.

[2] M. ABATE, "Iteration Theory of Holomorphic Maps on Taut Manifolds", Research and Lecture Notes in Mathematics, Complex Analysis and Geometry, Mediterranean Press, Rende 1989.

[3] M. ABATE, *Iteration theory, compactly divergent sequences and commuting holomorphic maps*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **18** (1991), 167–191.

[4] M. ABATE and J. RAISSY, *Wolff–Denjoy theorems in nonsmooth convex domains*, Ann. Mat. Pura Appl. **193** (2014), 1503–1518.

[5] W. BALLMANN, M. GROMOV and V. SCHROEDER, "Manifolds of Nonpositive Curvature", Progress in Mathematics, Vol. 61, Birkhäuser, Boston, MA, 1985.

[6] Z. M. BALOGH and M. BONK, *Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains*, Comment. Math. Helv. **75** (2000), 504–533.

[7] A. F. BEARDON, *Iteration of contractions and analytic maps*, J. London Math. Soc. **41** (1990), 141–150.

[8] A. F. BEARDON, *The dynamics of contractions*, Ergodic Theory Dynam. Systems **17** (1997), 1257–1266.
[9] G. BHARALI and A. ZIMMER, *Goldilocks domains, a weak notion of visibility, and applications*, Adv. Math. **310** (2017), 377–425.
[10] M. R. BRIDSON and A. HAEFLIGER, "Metric Spaces of Non-positive Curvature", Grundlehren der Mathematischen Wissenschaften, Vol. 319, Springer-Verlag, Berlin, 1999.
[11] M. BUDZYŃSKA, *The Denjoy–Wolff theorem in* $\mathbb{C}^n$, Nonlinear Anal. **75** (2012), 22–29.
[12] J. P. D'ANGELO, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. **115** (1982), 615–637.
[13] A. DENJOY, *Sur l'itération des fonctions analytiques*, C.R. Acad. Sci. Paris **182** (1926), 255–257.
[14] P. EBERLEIN and B. O'NEILL, *Visibility manifolds*, Pacific J. Math. **46** (1973) 45–109.
[15] F. FORSTNERIC and J.-P. ROSAY, *Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings*, Math. Ann. **279** (1987), 239–252.
[16] M. HERVÉ, *Quelques propriétés des applications analytiques d'une boule à m dimensions dan elle-même*, J. Math. Pures Appl. **42** (1963), 117–147.
[17] L. HORMANDER, "An Introduction to Complex Analysis in Several Variables", North-Holland Mathematical Library, North-Holland, Elsevier Science Publishers B.V., Amsterdam 1990.
[18] X. J. HUANG, *A non-degeneracy property of extremal mappings and iterates of holomorphic self-mappings*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), 399–419.
[19] M. JARNICKI and P. PFLUG, "Invariant Distances and Metrics in Complex Analysis", de Gruyter Expositions in Mathematics, Vol. 9, Walter de Gruyter & Co., Berlin, 1993.
[20] A. KARLSSON, *Non-expanding maps and Busemann functions*, Ergodic Theory Dynam. Systems **21** (2001), 1447–1457.
[21] N. KERZMAN and J.-P. ROSAY, *Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut*, Math. Ann. **257** (1981), 171–184.
[22] P. KIERNAN, *On the relations between taut, tight and hyperbolic manifolds*, Bull. Amer. Math. Soc. **76** (1970), 49–51.
[23] D. MA, *Sharp estimates of the Kobayashi metric near strongly pseudoconvex points*, In: "The Madison Symposium on Complex Analysis (Madison, WI, 1991)", Contemp. Math., Vol. 137, Amer. Math. Soc., Providence, RI, 1992, 329–338.
[24] H. L. ROYDEN, *Remarks on the Kobayashi metric*, in Several Complex Variables, II (Proc. Internat. Conf., Univ. Maryland, College Park, MD, 1970)", Lecture Notes in Math., Vol. 185, Springer, Berlin, 1971, 125–137.
[25] N. SIBONY, *A class of hyperbolic manifolds*, In: "Recent Developments in Several Complex Variables (Proc. Conf., Princeton Univ., Princeton, NJ, 1979)", Ann. of Math. Stud., Vol. 100, Princeton Univ. Press, Princeton, NJ, 1981, 357–372.
[26] J. WOLFF, *Sur une généralisation d'un théorème de Schwarz*, C.R. Acad. Sci. Paris **182** (1926), 918–920.
[27] H. WU, *Normal families of holomorphic mappings*, Acta Math. **119** (1967), 193–233.
[28] A. M. ZIMMER, *Gromov hyperbolicity and the Kobayashi metric on convex domains of finite type*, Math. Ann. **365** (2016), 1425–1498.
[29] A. M. ZIMMER, *Characterizing domains by the limit set of their automorphism group*, Adv. Math. **308** (2017), 438–482.

Department of Mathematics
Indian Institute of Science
Bangalore 560012, India
bharali@iisc.ac.in
anwoymaitra@iisc.ac.in